

MAT4510, Fall 2024

Solutions to assignment.

All references apply to the lecture notes.

Problem 1 Given an integer $n \geq 2$, let

$$L := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x, y \text{ are linearly dependent}\}.$$

(a) Show that $L' := L - \{(0, 0)\}$ is a manifold and determine its dimension.

Solution: Let $W := \mathbb{R}^n - \{0\}$ and define

$$F : W \times \mathbb{R} \rightarrow (W \times \mathbb{R}^n) \cap L', \quad (p, t) \mapsto (p, tp).$$

Then F is smooth and bijective, and the inverse map

$$F^{-1}(x, y) = \left(x, \frac{x \cdot y}{x \cdot x}\right)$$

is also smooth. Hence, F is a diffeomorphism. Similarly one can show that $(\mathbb{R}^n \times W) \cap L'$ is diffeomorphic to $W \times \mathbb{R}$. Since $W \times \mathbb{R}^n$ and $W \times \mathbb{R}^n$ are open subsets of $\mathbb{R}^n \times \mathbb{R}^n$ and

$$L' \subset (W \times \mathbb{R}^n) \cup (W \times \mathbb{R}^n),$$

we conclude that L' is a manifold of dimension $n + 1$.

(b) Show that L is not a manifold.

Solution: Suppose L is a manifold. Let $x \in \mathbb{R}^n$ and define

$$\gamma : \mathbb{R} \rightarrow L, \quad t \mapsto (tx, 0).$$

Then $\gamma'(0) = (x, 0)$, hence $(x, 0) \in T_{(0,0)}L$. Since this holds for any $x \in \mathbb{R}^n$, we conclude that $\mathbb{R} \times \{0\} \subset T_{(0,0)}L$. In the same way one shows that $\{0\} \times \mathbb{R} \subset T_{(0,0)}L$. Therefore, the tangent space of L at $(0, 0)$ is all of $\mathbb{R}^n \times \mathbb{R}^n$, hence $\dim L = 2n$. This means that L is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. However, any neighbourhood of $(0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^n$ contains points outside L , a contradiction. (We could also have invoked the result of part (a) to obtain a contradiction.) Hence, L is not a manifold.

Problem 2 Let

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)z^2 = 1, z > 0\}.$$

(a) Show that S is a regular surface.

Solution: S is the graph of the function

$$\mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto (x^2 + y^2)^{-1/2},$$

hence a regular surface.

(b) Compute the Gauss curvature of S .

Solution: Using polar coordinates in the xy -plane we have

$$S = \{(\rho \cos \theta, \rho \sin \theta, z) \mid \rho, \theta \in \mathbb{R}, \rho > 0, \rho z = 1\}.$$

Therefore, S is the surface obtained by revolving the 1-manifold

$$C := \{(\rho, z) \in \mathbb{R}^2 \mid \rho > 0, \rho z = 1\}$$

around the z -axis. We consider the global parametrization

$$\gamma : \mathbb{R}_+ \rightarrow C, \quad t \mapsto (t, t^{-1}).$$

If F is a local parametrization of S is in Section 2.5 then the formula (2.5) for the Gauss curvature K of S obtained there yields

$$K(F(t, \theta)) = -\frac{2t^2}{(t^4 + 1)^2}.$$

Problem 3 Let $S \subset \mathbb{R}^3$ be a regular surface and $r > 0$. Define

$$\bar{S} := \{rp \mid p \in S\}.$$

Let K and $\bar{K}$ be the Gauss curvatures of S and $\bar{S}$, respectively. Express $\bar{K}$ in terms of K .

Solution: The map

$$\phi : S \rightarrow \bar{S}, \quad p \mapsto rp$$

is a diffeomorphism and $d\phi_p(v) = rp$. This shows that

$$T_{rp}\bar{S} = T_p S$$

for all $p \in S$. The inverse map

$$\psi := \phi^{-1} : \bar{S} \rightarrow S$$

is given by $\psi(q) = q/r$. Let $V \subset S$ be an open subset on which there exists a smooth unit normal field $N : V \rightarrow S^2$. On $\bar{V} := \phi(V)$ we have a smooth unit normal field $\bar{N}$ given by

$$\bar{N}_q = N_{q/r},$$

for $q \in \bar{V}$, i.e. $\bar{N} = N \circ \psi|_{\bar{V}}$. For $q = rp$,

$$d\bar{N}_q(w) = dN_p \circ d\psi_q(w) = dN_p(r^{-1}w) = r^{-1}dN_p(w).$$

Therefore,

$$\bar{K}(q) = \det(d\bar{N}_q) = r^{-2} \det(dN_p) = r^{-2}K(p).$$

In other words,

$$\bar{K}(rp) = r^{-2}K(p)$$

for all $p \in S$.

Problem 4 A smooth vector field Y on a manifold M is called *parallel* if $\nabla_X Y = 0$ for all vector fields X on M . Prove the following.

(a) If Y is parallel then the function $p \mapsto \|Y_p\|$ on M is locally constant.

Solution: For every vector field X on M ,

$$\nabla_X \langle Y, Y \rangle = 2 \langle \nabla_X Y, Y \rangle = 0.$$

By the result of Problem 3.4, the function $\|Y\|$ on M is locally constant.

(b) If a regular surface S admits a non-zero parallel vector field Y then the Gauss curvature of S vanishes.

Hint: By rescaling Y we obtain a parallel vector field E_1 of unit length. Locally, one can extend E_1 to a smooth orthonormal frame (E_1, E_2) . Prove

that E_2 is also parallel by considering $\partial_X \langle E_i, E_j \rangle$ for arbitrary vector fields X .

Solution: Let $E_1 := Y/\|Y\|$. Since $\|Y\|$ is locally constant,

$$\nabla_X E_1 = \|Y\|^{-1} \nabla_X Y = 0$$

for every vector field X , so E_1 is parallel.

Given $p \in S$, we can find a smooth vector field Z on S such that $E_1(p)$ and $Z(p)$ are linearly independent, or equivalently, the cross product $E_1 \times Z$ is non-zero at p . By continuity, $E_1(q)$ and $Z(q)$ are linearly independent for all q in some neighbourhood V of p in S . Applying the Gram-Schmidt process pointwise to the smooth frame (E_1, Z) on V yields a smooth orthonormal frame (E_1, E_2) on V . If $S \subset \mathbb{R}^3$ then it now follows from Theorem 3.7.2 that the Gauss curvature K of S vanishes. If $S \subset \mathbb{R}^n$ with $n > 3$ then K was defined by Equation (3.5), so $K = 0$ in this case, too.

We add a proof that E_2 is parallel, although this was not needed for the proof that $K = 0$. For any vector field X on V ,

$$0 = \partial_X \langle E_i, E_j \rangle = \langle \nabla_X E_i, E_j \rangle + \langle E_i, \partial_X E_j \rangle$$

for all i, j . Therefore,

$$\begin{aligned} \langle \nabla_X E_2, E_1 \rangle &= -\langle E_2, \nabla_X E_1 \rangle = 0, \\ \langle \nabla_X E_2, E_2 \rangle &= \frac{1}{2} \partial_X \langle E_2, E_2 \rangle = 0, \end{aligned}$$

so $\nabla_X E_2 = 0$, i.e. E_2 is parallel.