Introduction to Differential Geometry

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Preface

These notes provide an introduction to differential geometry aimed at first year graduate students. Only submanifolds of finite-dimensional Euclidean spaces are considered, and the emphasis is on surfaces. Except in the final chapter on hyperbolic geometry, the only Riemannian metric considered on a submanifold of $\mathbb{R}^n$ is the one induced from the usual scalar product on $\mathbb{R}^n$.

Parts of the text have been influenced by the book [1] of Ch. Bär, whereas the version of Gauss-Bonnet's theorem proved here is taken from B. Jahren's book [6], with no assumption on the orientability of the surface.

Readers are assumed to have a basic knowledge of multivariable calculus and linear algebra. Readers not already familiar with basic point-set topology should study this subject concurrently.

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Chapter 1

Manifolds

In this chapter we introduce submanifolds of Euclidean spaces (just referred to as manifolds) and fundamental notions such as tangent spaces, smooth maps, and vector fields.

1.1 Basic notions

For sets A and B, we write $A \subset B$ to express that A is a subset of B, i.e. that every element of A is also an element of B. This does not rule out that $A = B$.

Given two points $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ in $\mathbb{R}^n$, their Euclidean scalar product (or inner product) is

$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i.$$

The Euclidean norm of x is defined by

$$\|x\| := \sqrt{\langle x, x \rangle} = \sum_{i=1}^{n} (x^i)^2.$$

We denote by $B_r^n(x)$ the Euclidean ball in $\mathbb{R}^n$ of radius $r > 0$ and centre x, i.e.

$$B_r^n(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}.$$

The **unit ball** is $B^n := B_1^n(0)$. A subset $U \subset \mathbb{R}^n$ is called **open** if for every $x \in U$ there exists an $r > 0$ such that $B_r^n(x) \subset U$.

Example 1.1.1 Each ball $B_r^n(x)$ is an open subset of $\mathbb{R}^n$.

If A and B are subsets of $\mathbb{R}^n$ such that $A \subset B$ then A is called an **open subset of** B provided there exists an open subset $U \subset \mathbb{R}^n$ such that $A = B \cap U$. In that case, we say A is **open in** B. If in addition $x \in A$ then A is called a **neighbourhood of** x **in** B.

Given subsets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, a map $F : A \to B$ is called a **homeomorphism** if F is bijective and both F and its inverse map $F^{-1} : B \to A$ are continuous.

If $U \subset \mathbb{R}^m$ is open and k is a non-negative integer, a function $F : U \to \mathbb{R}^n$ is said to be of **class** C^k or k **times continuously differentiable** if all partial derivatives of F or order $\leq k$ exist and are continuous. (In particular, F is of class C^0 if and only if it is continuous.) If F is of class C^k for all $k \geq 0$ then F is said to be of **class** C^∞, **smooth**, or **infinitely differentiable**.

Given subsets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, a map $F : A \to B$ is said to be **smooth** if for every point $x \in A$ there exist a neighbourhood U of x in $\mathbb{R}^m$ and a map $\tilde{F} : U \to \mathbb{R}^n$ which is smooth in the sense already defined, and such that $F = \tilde{F}$ on $A \cap U$. If in addition F is bijective and $F^{-1} : B \to A$ is smooth as well then F is called a **diffeomorphism**. The subsets A and B are called **diffeomorphic** if there exists a diffeomorphism $A \to B$. Note that every smooth map $A \to B$ is continuous.

Example 1.1.2 Let

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

be the open unit ball in $\mathbb{R}^n$ and consider the maps

$$B^n \xrightarrow{F} \mathbb{R}^n \xrightarrow{G} B^n$$

given by

$$F(x) = \frac{x}{\sqrt{1 - \|x\|^2}}, \quad G(y) = \frac{y}{\sqrt{1 + \|y\|^2}}.$$

Then F and G are smooth, and a simple computation shows that they are inverse maps, hence diffeomorphisms.

Example 1.1.3 If $A \subset B \subset \mathbb{R}^n$ then the inclusion map $A \to B$ is smooth.

Proposition 1.1.1 *Suppose*

$$A \xrightarrow{F} B \xrightarrow{G} C$$

are smooth maps, where $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, and $C \subset \mathbb{R}^p$. Then the composite map $G \circ F : A \to C$ is smooth.

Proof. If A and B are open in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, then the rules of differentiation from calculus easily implies that $G \circ F$ is smooth. For the general case, let $x \in A$. Choose a neighbourhood U of x in $\mathbb{R}^m$ and a smooth map $\tilde{F} : U \to \mathbb{R}^n$ which agrees with F on $A \cap U$. Also, choose a neighbourhood V of $F(x)$ in $\mathbb{R}^n$ and a smooth map $\tilde{G} : V \to \mathbb{R}^p$ which agrees with G on $B \cap V$. Because $\tilde{F}$ is continuous, the preimage $U_0 := \tilde{F}^{-1}V$ is open in U, hence U_0 is also open in $\mathbb{R}^m$ (since U is open in $\mathbb{R}^m$). Moreover, $x \in U_0$. Now,

$$\tilde{G} \circ \tilde{F} : U_0 \to \mathbb{R}^n$$

is a smooth map which agrees with $G \circ F$ on $A \cap U_0$. $\square$

1.2 Manifolds

If k is a non-negative integer, a subset $M \subset \mathbb{R}^n$ is called a k–**dimensional manifold** (or just k–**manifold**) if every point in M has a neighbourhood V in M which is diffeomorphic to an open subset U of $\mathbb{R}^k$. A diffeomorphism $\phi : V \to U$ is called a **chart** on M, and the inverse map $F := \phi^{-1} : U \to V$ is called a **local parametrization** of M. For any $p \in V$, the map F is called a local parametrization of M **at** p. If $p = F(u)$ then the components $u^1, \ldots, u^k$ of u are called the **coordinates** of p with respect to the parametrization F. The image $V = F(U)$ is called a **coordinate domain**. If $V = M$ then F is called a **global parametrization** . The integer k is called the **dimension** of M and is denoted by $\dim M$. If M has dimension $n - 1$ then it is called a **hypersurface** in $\mathbb{R}^n$. A 1–manifold is also called a **regular curve**, and a 2–manifold is called a **regular surface**.

The above definition of a manifold is taken from the classical text [4]. For the definition of an abstract manifold (not necessarily a subset of any Euclidean space), see [14, 9].

If M and N are manifolds and $M \subset N$ then M is called a **submanifold** of N, and the **codimension** of M in N is

$$\operatorname{codim}_N M = \dim N - \dim M.$$

A smooth map $f : M \to N$ is called a **smooth embedding** if $f(M)$ is a manifold and the map

$$M \to f(M), \quad p \mapsto f(p)$$

is a diffeomorphism.

Example 1.2.1 For $n \geq 0$ the n–**sphere** S^n is defined by

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

We will show that S^n is an n–manifold. The open half-space

$$H := \{(x^0, x^1, \ldots, x^n) \in \mathbb{R}^n \mid x_0 > 0\}$$

intersect S^n in the open hemisphere $V := S^n \cap H$. Forgetting the first coordinate yields a smooth map

$$H \to \mathbb{R}^n, \quad (x^0, x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n)$$

which restricts to a smooth map

$$\phi : V \to B^n.$$

The map ϕ is in fact bijective with inverse map

$$\phi^{-1}(y) = (\sqrt{1 - \|y\|^2}, y^1, \ldots, y^n).$$

Since ϕ^{-1} is smooth as well, it follows that ϕ is a diffeomorphism and therefore a chart on S^n. To produce more charts, observe that any orthogonal $(n+1) \times (n+1)$ matrix A defines a diffeomorphism

$$\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad x \mapsto Ax$$

which restricts to a diffeomorphism $T_A : S^n \to S^n$. Let

$$V_A := \{x \in S^n \mid Ax \in V\},$$

which is an open subset of S^n. Then the composite map

$$\phi \circ T_A : V_A \to B^n$$

is a diffeomorphism. As A varies, the sets V_A will cover all of S^n, proving that S^n is an n–manifold. $\square$

Proposition 1.2.1 *If $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are manifolds, then the product $M \times N \subset \mathbb{R}^{m+n}$ is also a manifold, and*

$$\dim (M \times N) = \dim M + \dim N.$$

Proof. See Problem 1.7. $\square$

Example 1.2.2 The product $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ is called the 2–**torus**, or just the **torus**.

1.3 Regular values

If $U \subset \mathbb{R}^m$ is an open set and $F : U \to \mathbb{R}^n$ a smooth map then $\partial_i F : U \to \mathbb{R}^n$ will denote the ith partial derivative of F. In other words, if $(u^1, \ldots, u^m)$ are the standard coordinates on $\mathbb{R}^m$ then

$$\partial_i F = \frac{\partial F}{\partial u^i}.$$

The **differential** (or **derivative**) of F at a point $u \in \mathbb{R}^m$ is the linear map $dF_u : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$dF_u(a) = \lim_{t \to 0} \frac{F(u + ta) - F(u)}{t}$$

for all $a \in \mathbb{R}^m$. If $a = (a^1, \ldots, a^m)$ then

$$dF_u(a) = \sum_{i=1}^{m} a^i \partial_i F(u).$$

Therefore, if F has components $F^1, \ldots, F^n$ then the basis for dF_u with respect to the standard bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ is the Jacobian matrix $(\partial_j F^i(u))$.

We now recall two theorems from calculus.

Theorem 1.3.1 (Chain rule, I) *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. If*

$$U \xrightarrow{F} V \xrightarrow{G} \mathbb{R}^p$$

are smooth maps then the differential of the composite map $G \circ F : U \to \mathbb{R}^p$ at a point $u \in U$ is given by

$$d(G \circ F)_u = dG_{F(u)} \circ dF_u. \qquad \square$$

Theorem 1.3.2 (Inverse function theorem, I) *Let $F : U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. If $p \in U$ is a point where the differential dF_p is an isomorphism then F maps some neighbourhood of p in U diffeomorphically onto a neighbourhood of $F(p)$ in V.* $\square$

Given an open set $\Omega \subset \mathbb{R}^n$ and a smooth map $F : \Omega \to \mathbb{R}^m$, a point $p \in \Omega$ is called a **regular point** for F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is surjective; otherwise p is called a **singular point**. A point $c \in \mathbb{R}^m$ is called a **regular value** for F if every point in $F^{-1}(c)$ is a regular point for F;

otherwise c is called a **singular value**. Note that if c does not lie in the image of F then it is a regular value. Sard's theorem (see [9]) says that the set of singular values of F has measure zero. From this it follows that the set of regular values is dense in $\mathbb{R}^m$.

Theorem 1.3.3 *If Ω is an open set in $\mathbb{R}^n$ and c is a regular value for $H : \Omega \to \mathbb{R}^m$ then the preimage $S := H^{-1}(c)$ is a manifold of dimension $n - m$.*

Proof. After replacing H by $H - c$ we may assume that $c = 0$. Let $p \in S$ and let K be the kernel of the differential

$$dH_0 : \mathbb{R}^n \to \mathbb{R}^m$$

of H at 0. Then K has dimension $n - m$. Choose a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^{n-m}$$

which restricts to an isomorphism $K \to \mathbb{R}^{n-m}$ and define

$$\Phi : \Omega \to \mathbb{R}^{n-m} \times \mathbb{R}^m, \quad x \mapsto (Ax, H(x)).$$

The differential of Φ at p is then given by

$$d\Phi_p(v) = (Av, dH_p(v)).$$

It follows that the kernel of $d\Phi_p$ is 0, hence $d\Phi_p$ is a linear isomorphism. By the inverse function theorem, Φ maps some neighbourhood U of p in Ω diffeomorphically onto an open set V in $\mathbb{R}^{n-m} \times \mathbb{R}^m$. By definition of Φ we have

$$\Phi(S \cap U) = (\mathbb{R}^{n-m} \times \{0\}) \cap V.$$

Let Φ have components $\Phi^1, \ldots, \Phi^m$ and let

$$\phi : \Omega \to \mathbb{R}^{n-m}$$

be the map with components $\Phi^1, \ldots, \Phi^{n-m}$. Then ϕ maps $S \cap U$ diffeomorphically onto the open subset of $\mathbb{R}^{n-m}$ consisting of all $y \in \mathbb{R}^{n-m}$ for which $(y, 0) \in V$. This shows that S is a manifold of dimension $n - m$. $\square$

Example 1.3.1 (Sphere) Let

$$H : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|x\|^2.$$

The gradient of H at x is $\nabla H_x = 2x$, which vanishes only for $x = 0$. Therefore, $0 \in \mathbb{R}^n$ is the only singular point of H, and $0 \in \mathbb{R}$ is the only singular value. Theorem 1.3.3 now confirms that the $(n-1)$–sphere

$$S^{n-1} = H^{-1}(1)$$

is an $(n-1)$–manifold.

Example 1.3.2 (Special linear group) Let $M_{n\times n}(\mathbb{R})$ be the set of all $n \times n$ real matrices. We will show that the special linear group

$$\mathrm{SL}_n(\mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$$

is a manifold of dimension (n^2-1). After fixing some ordering of the entries of $n \times n$ matrices, we can identify $M_{n\times n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$. The determinant

$$H : M_{n\times n}(\mathbb{R}) \to \mathbb{R}, \quad A \mapsto \det(A)$$

is a polynomial function and therefore smooth. For any $n \times n$ matrix A and real number t we have

$$\det(tA) = t^n \det(A).$$

Differentiating this equation with respect to t and applying the chain rule we get

$$dH_A(A) = \left.\frac{d}{dt}\right|_{t=1} \det(tA) = n \det(A).$$

If $\det(A) \neq 0$ then A is therefore a regular point for H. By Theorem 1.3.3, the preimage

$$\mathrm{SL}_n(\mathbb{R}) = H^{-1}(1)$$

is an (n^2-1)–manifold.

1.4 Tangent spaces

Let $M \subset \mathbb{R}^n$ be a k–manifold and $p \in M$. A vector $v \in \mathbb{R}^n$ is called a **tangent vector** of M at p if there exist an $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that

$$\gamma(0) = p, \quad \gamma'(0) = v,$$

where γ' is the calculus derivative of γ (or more precisely, the derivative of $\iota \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, where $\iota : M \to \mathbb{R}^n$ is the inclusion map). The set T_pM of all tangent vectors to M at p is called the **tangent space** of M at p.

Proposition 1.4.1 *If $F : U \to M$ is a local parametrization of the k–manifold M then for any $u \in U$, the differential dF_u maps $\mathbb{R}^k$ isomorphically onto the tangent space $T_{F(u)}M$.*

Proof. Let $M \subset \mathbb{R}^n$, $V := F(U)$, and $p := F(u)$. Let $\phi := F^{-1} : V \to U$. We show that

- the differential $dF_u : \mathbb{R}^k \to \mathbb{R}^n$ is injective, and
- the image of dF_u is equal to the tangent space T_pM.

(i) We first show that dF_u is injective. By assumption, there exist a neighbourhood W of p in $\mathbb{R}^n$ and a smooth map $\Phi : W \to \mathbb{R}^k$ which agrees with ϕ on $V \cap W$. Then $U_0 := F^{-1}W$ is a neighbourhood of u in $\mathbb{R}^k$, and $\Phi \circ F = \mathrm{Id}$ on U_0. By the chain rule we have $d\Phi_p \circ dF_u = \mathrm{Id}_{\mathbb{R}^k}$, which shows that dF_u is injective.

(ii) Let $v \in T_pM$. We will show that $v \in \mathrm{im}(dF_u)$. Choose a curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. After perhaps shrinking ϵ we may assume that γ takes values in V. Then $\gamma = F \circ (\phi \circ \gamma)$, so by the chain rule we have

$$\gamma'(0) = dF_u((\phi \circ \gamma)'(0)) \in \mathrm{im}(dF_u).$$

This shows that $T_pM \subset \mathrm{im}(dF_u)$. To prove the reverse inclusion, let $a \in \mathbb{R}^k$. Choose $\epsilon > 0$ such that

$$u + ta \in U \quad \text{for } |t| < \epsilon.$$

Define $\gamma : (-\epsilon, \epsilon) \to M$ by

$$\gamma(t) = F(u + ta).$$

Then $dF_u(a) = \gamma'(0) \in T_pM$. $\square$

Corollary 1.4.1 *If $M \subset \mathbb{R}^n$ is a k–manifold then for any $p \in M$, the tangent space T_pM is a k–dimensional linear subspace of $\mathbb{R}^n$.* $\square$

We now extend the notion of differentials to maps between manifolds.

Proposition 1.4.2 *If $F : M \to N$ is a smooth map between manifolds then for any $p \in M$ there is a unique map*

$$dF_p : T_pM \to T_{F(p)}N$$

such that for any curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ one has

$$dF_p(\gamma'(0)) = (F \circ \gamma)'(0).$$

Moreover, dF_p is linear.

Proof. Uniqueness is obvious. To prove existence, suppose $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$. Choose a neighbourhood W of p in $\mathbb{R}^m$ and a smooth map $\tilde{F} : W \to \mathbb{R}^n$ which agrees with F on $M \cap W$. We define dF_p to be the restriction of the differential $d\tilde{F}_p : \mathbb{R}^m \to \mathbb{R}^n$ to T_pM. Then dF_p is linear. Now let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\gamma(0) = p$. After perhaps shrinking ϵ we may assume that γ takes values in $M \cap W$. By the chain rule,

$$d\tilde{F}_p(\gamma'(0)) = (\tilde{F} \circ \gamma)'(0) = (F \circ \gamma)'(0),$$

proving that dF_p has the desired properties. $\square$

The map dF_p is called the **differential** or **tangent map** of F at p.

Example 1.4.1 Let

$$\mathrm{Id}_M : M \to M, \quad p \mapsto p$$

be the identity map. Then for any $p \in M$, the differential $d(\mathrm{Id}_M)_p$ is the identity map on the tangent space T_pM, i.e. for all $v \in T_pM$ one has

$$d(\mathrm{Id}_M)_p(v) = v.$$

Proposition 1.4.3 (Chain rule, II) *If $M \xrightarrow{F} N \xrightarrow{G} P$ are smooth maps between manifolds and $p \in M$, $q = F(p)$ then*

$$dG_q \circ dF_p = d(G \circ F)_p.$$

Proof. If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve with $\gamma(0) = p$ then

$$\begin{aligned} d(G \circ F)_p(\gamma'(0)) &= (G \circ F \circ \gamma)'(0) \\ &= dG_q((F \circ \gamma)'(0)) \\ &= (dG_q \circ dF_p)(\gamma'(0)). \qquad \square \end{aligned}$$

Corollary 1.4.2 *If $F : M \to N$ is a diffeomorphism then for every $p \in M$ the differential dF_p is an isomorphism.*

Proof. Let $G := F^{-1} : N \to M$. Let $p \in M$ and $q := F(p)$. Since $G \circ F = \mathrm{Id}_M$, the chain rule yields

$$dG_q \circ dF_p = d(G \circ F)_p = d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM},$$

and similarly $dF_p \circ dG_q = \mathrm{Id}_{T_qM}$. Hence, dF_p is an isomorphism. $\square$

Theorem 1.4.1 (Inverse function theorem, II) *Let $H : M \to N$ be a smooth map between manifolds. If $p \in M$ is a point where the differential dH_p is an isomorphism then H maps some neighourhood of p in M diffeomorphically onto a neighbourhood of $H(p)$ in N.*

Proof. Since dH_p is an isomorphism, the manifolds M and N must have the same dimension, say $n := \dim M = \dim N$. Let $q := H(p)$. Choose local parametrizations $F : U \to M$ at p and $G : V \to N$ at q, so that U and V are open subsets of $\mathbb{R}^n$. Let $u := F^{-1}(p)$ and $v := G^{-1}(q)$. Since $G(V)$ is open in N and $H \circ F$ is continuous, the preimage $U_0 := (H \circ F)^{-1}(G(V))$ is open in U. Restricting F to U_0 we obtain a local parametrization

$$F_0 := F|_{U_0} : U_0 \to M.$$

Applying the chain rule to the composite map

$$\tilde{H} := G^{-1} \circ H \circ F_0 : U_0 \to V$$

we get

$$d\tilde{H}_u = d(G^{-1})_q \circ dH_p \circ d(F_0)_u.$$

Thus, $d\tilde{H}_u$ is a composite of three isomorphism, hence an isomorphism. By Theorem 1.3.2, $\tilde{H}$ maps some open subset $U_1 \subset U_0$ diffeomorphically onto an open subset $V_1 \subset V$. But then $F(U_1)$ is a neighbourhood of p in M which is mapped diffeomorphically by H onto the neighbourhood $G(V_1)$ of q in N. □

A smooth map $\phi : M \to M'$ between manifolds is called a **local diffeomorphism** if every point in M has a neighbourhood V in M such that $\phi(V)$ is open in M' and $\phi|_V : V \to \phi(V)$ is a diffeomorphism.

Proposition 1.4.4 *For any smooth map $\phi : M \to M'$ between manifolds the following are equivalent.*

(i) *ϕ is a local diffeomorphism.*

(ii) *For every $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}M'$ is an isomorphism.*

Proof. This follows from Corollary 1.4.2 and the inverse function theorem, Theorem 1.4.1. □

Proposition 1.4.5 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F : \Omega \to \mathbb{R}^m$ a smooth map. If $c \in \mathbb{R}^m$ is a regular value for F and $S = F^{-1}(c)$, $p \in S$, then*

$$T_pS = \ker(dF_p).$$

Proof. If $\gamma : (-\epsilon, \epsilon) \to X$ is a smooth curve with $\gamma(0) = p$ then $F \circ \gamma \equiv c$, hence

$$0 = (F \circ \gamma)'(0) = dF_p(\gamma'(0)).$$

This shows that $T_pS \subset \ker(dF_p)$. But this is an inclusion between vector spaces of the same dimension (namely $n-m$), so the two vector spaces must be equal. $\square$

Note that if $m = 1$, so that F is a real-valued function on Ω, then the differential of F at a point $u \in \Omega$ is given by

$$dF_u(a) = \langle \nabla F_u, a \rangle,$$

where ∇F_u is the gradient of F at u. In this case, the tangent space of S at a point $p \in S$ is the orthogonal complement of ∇F_p in $\mathbb{R}^n$.

Example 1.4.2 Let

$$F : \mathbb{R}^n \to \mathbb{R}, \quad u \mapsto \|u\|^2.$$

The tangent space to the sphere $S^{n-1} = F^{-1}(1)$ at a point u is the orthogonal complement of the gradient $\nabla F_u = 2u$, i.e.

$$T_uS^{n-1} = \{a \in \mathbb{R}^n \mid \langle u, a \rangle = 0\}.$$

Example 1.4.3 We will compute the tangent space to the special linear group $\mathrm{SL}_n(\mathbb{R})$ at the identity matrix I. Let

$$F = \det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}.$$

For any real number t and $A \in M_{n\times n}(\mathbb{R})$ one has

$$\det(I + tA) = 1 + \mathrm{tr}(A) \cdot t + \cdots + \det(A) \cdot t^n.$$

Therefore,

$$dF_I(A) = \left.\frac{d}{dt}\right|_0 \det(I + tA) = \mathrm{tr}(A),$$

from which we conclude that the tangent space to $\mathrm{SL}_n(\mathbb{R})$ at I consists of the trace-free matrices, i.e.

$$T_I\mathrm{SL}_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}.$$

1.5 Vector fields and normal fields

Let $M \subset \mathbb{R}^n$ be a k–manifold. A **vector field** on M is a map $X : M \to \mathbb{R}^n$ such that

$$X(p) \in T_pM \quad \text{for all } p \in M.$$

One often writes X_p instead of $X(p)$. A **frame** on an open set $V \subset M$ is a k–tuple $(X_1, \ldots, X_k)$ of vector fields on V such that $(X_1(p), \ldots, X_k(p))$ is a basis for the tangent space T_pM for every $p \in V$. The frame is called **continuous** (or **smooth**) if each X_i is continuous (or smooth). By a **local frame** on M we mean a frame on some open subset of M.

Example 1.5.1 Any local parametrization $F : U \to M$ gives rise to **coordinate vector fields** $X_1, \ldots, X_k$ on $V := F(U)$ satisfying

$$X_i \circ F = \partial_i F.$$

Thus, if $u \in U$ and $p = F(u)$ then $X_i(p) = \partial_i F(u)$. Each vector field X_i is smooth, since it is the composition of the two smooth maps

$$V \xrightarrow{F^{-1}} U \xrightarrow{\partial_i F} \mathbb{R}^n.$$

If $p = F(u)$ then the derivative dF_u maps the standard basis for $\mathbb{R}^k$ to the basis $(X_1(p), \ldots, X_k(p))$ for T_pM, hence $(X_1, \ldots, X_k)$ is a smooth frame on V, called a **coordinate frame**.

Example 1.5.2 Any skew-symmetric $n \times n$ real matrix A gives rise to a smooth vector field X on the sphere S^{n-1} defined as follows. Let

$$X : S^{n-1} \to \mathbb{R}^n, \quad p \mapsto Ap.$$

Because $A^T = -A$ we have

$$\langle Ap, p\rangle = -\langle p, Ap\rangle = -\langle Ap, p\rangle,$$

so $\langle Ap, p\rangle = 0$. Therefore, $X(p) \in T_pS^{n-1}$, so X is a vector field on S^{n-1}.

A **normal field** on a manifold $M \subset \mathbb{R}^n$ is a map

$$N : M \to \mathbb{R}^n$$

such that $N(p)$ is perpendicular to T_pM for every $p \in M$. If in addition $\|N(p)\| = 1$ for every p then N is called a **unit normal field**. One often writes N_p instead of $N(p)$.

Example 1.5.3 Let $S \subset \mathbb{R}^3$ be a regular surface. If X_1, X_2 are the coordinate vector fields corresponding to a local parametrization $F : U \to S$ then, for every $p \in F(U)$, the cross product $X_1(p) \times X_2(p)$ is non-zero and perpendicular to the tangent space T_pS. Therefore, we can define a smooth unit normal field N on $F(U)$ by

$$N(p) := \frac{X_1(p) \times X_2(p)}{\|X_1(p) \times X_2(p)\|}.$$

Example 1.5.4 Let $\Omega \subset \mathbb{R}^n$ be an open set and $F : \Omega \to \mathbb{R}$ a smooth function. Suppose $c \in \mathbb{R}$ is a regular value for F, so that the preimage $S := F^{-1}(c)$ is an $(n-1)$–manifold. Then the gradient of F restricts to a smooth normal field N on S, i.e.

$$N : S \to \mathbb{R}^n, \quad p \mapsto \nabla F_p.$$

1.6 Orientations

Let V be a k–dimensional real vector space, $k > 0$. Two ordered bases $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ for V are **equivalent** if the linear isomorphism $V \to V$ mapping each u_i to v_i has positive determinant. This is an equivalence relation on the set of ordered bases for V. An **orientation** of V is a choice of equivalence class of ordered bases. Note that V has exactly two orientations. Given an orientation μ of V, the pair (V, μ) is called an **oriented vector space**. An ordered basis α for V is **positively oriented** if $\alpha \in \mu$, otherwise it is **negatively oriented**.

An orientation of a zero-dimensional vector space is defined to be a choice of sign ± 1.

An **orientation** of a k–manifold M is a choice of orientation of each tangent space T_pM, $p \in M$, such that the following continuity condition holds. For every $p \in M$ there should exist a neighbourhood W of p in M and continuous vector fields $X_1, \ldots, X_k$ on W such that $(X_1(q), \ldots, X_k(q))$ is a positively oriented basis for the tangent space T_qM for every $q \in W$. If M is oriented, then a frame $(X_1, \ldots, X_k)$ on an open set $V \subset M$ is called **positively oriented** if, for every $p \in V$, the k–tuple $(X_1(p), \ldots, X_k(p))$ is a positively oriented basis for T_pM.

The **standard orientation** for $\mathbb{R}^k$ is specified by the standard ordered basis $(e_1, \ldots, e_k)$. This orients every open subset $U \subset \mathbb{R}^k$, since $T_pU = \mathbb{R}^k$ for every $p \in U$.

A diffeomorphism $F : M \to M'$ between oriented manifolds is called **orientation preserving** if for every $p \in M$ the differential $dF_p : T_pM \to T_{F(p)}M'$ maps positively oriented ordered bases to positively oriented ones. In particular, this notion applies to local parametrizations of an oriented manifold.

Lemma 1.6.1 *Let M be an oriented manifold and $p \in M$. Then there exists a positively oriented smooth frame on some neighbourhood of p in M.*

Proof. By assumption, there exists a positively oriented continuous frame $(Y_1, \ldots, Y_k)$ on some neighbourhood V_0 of p in M. Choose any smooth frame $(X_1, \ldots, X_k)$ (for instance a coordinate frame) on some neighbourhood V_1 of p in M. Then $V := V_0 \cap V_1$ is also a neighbourhood of p. Given $q \in V$, each tangent vector $Y_j(q)$ can be expressed as a linear combination of the basis $(X_1(q), \ldots, X_k(q))$ for T_qM. This gives rise to a $k \times k$ matrix (a^i_j) of real-valued function on V such that

$$Y_j = \sum_i a^i_j X_i.$$

The matrix $(a^i_j(q))$ is invertible for every $q \in V$, being the transition matrix of two bases for T_qM. By the result of Problem 1.11, each function a^i_j is continuous, hence the determinant

$$\det((a^i_j)) : V \to \mathbb{R} - \{0\}$$

is continuous. Therefore, the sign of this determinant must be constant in some neighbourhood $V' \subset V$ of p in M. By replacing X_1 by $-X_1$ if necessary, we may arrange that the sign is positive in V'. But then $(X_1, \ldots, X_k)$ is a positively oriented smooth frame on V', and the lemma is proved. □

Proposition 1.6.1 *For $n \geq 1$, a hypersurface $S \subset \mathbb{R}^n$ is orientable if and only if it has a smooth unit normal vector field.*

For $n = 1$ the proposition is trivial. As for $n = 2$, one can show that every 1–manifold is orientable.

Proof. For simplicity, we will write out the proof only for $n = 3$. The general case is similar, see Problem 1.13.

(i) Suppose S is oriented. We define a normal vector field N on S as follows. For every $p \in S$ let $N(p) \in \mathbb{R}^3$ be the unique unit normal vector to the tangent space T_pS such that

$$\det(\zeta_1, \zeta_2, N(p)) > 0 \tag{1.1}$$

for some (hence every) positively oriented ordered basis (ζ_1, ζ_2) for T_pS. We show that N is smooth. Given $p \in M$, Lemma 1.6.1 ensures that we can find a positively oriented smooth frame (X_1, X_2) on some neighbourhood V of p in M. We can now define a smooth unit normal field N' on V by

$$N' := \frac{X_1 \times X_2}{\|X_1 \times X_2\|}.$$

Then

$$\det(X_1, X_2, N') = \langle X_1 \times X_2, N' \rangle = \|X_1 \times X_2\| > 0,$$

which shows that $N' = N$ on V. Hence, N is smooth.

(ii) Conversely, suppose N is a smooth unit normal vector field on S. We define an orientation of S by decreeing an ordered basis (ζ_1, ζ_2) for T_pS to be positively oriented if the condition (1.1) holds. We now verify that the continuity condition for an orientation is satisfied. If X_1, X_2 are coordinate vector fields on a connected open set $V \subset S$ then

$$\det(X_1(p), X_2(p), N(p)) \neq 0 \quad \text{for all } p \in V.$$

Because V is connected, the sign must be the same for all p. By perhaps interchanging X_1, X_2 we can arrange that

$$\det(X_1, X_2, N) > 0 \quad \text{on } V.$$

But this means that $(X_1(p), X_2(p))$ is a positively oriented basis for T_pS for every V. This shows that the continuity condition is satisfied. $\square$

Example 1.6.1 Let $\Omega \subset \mathbb{R}^m$ be an open set and $f : \Omega \to \mathbb{R}$ a smooth function. If $c \in \mathbb{R}$ is a regular value for f, then by Theorem 1.3.3 the preimage $S := f^{-1}(c)$ is a hypersurface in $\mathbb{R}^m$. For every $p \in S$ the gradient ∇f_p is non-zero and normal to the tangent space T_pS, so we can define a smooth unit normal field N on S by

$$N_p := \frac{\nabla f_p}{\|\nabla f_p\|}.$$

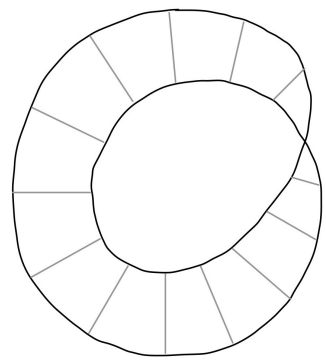
This shows that S is orientable.

Example 1.6.2 (Möbius strip) Topologically, the Möbius strip (without boundary) is obtained from the rectangle $[0, 1] \times (-1, 1)$ by identifying each point $(0, t)$ with $(1, -t)$, where $-1 < t < 1$, see Figure 1.1.

As a manifold, the Möbius strip can be conveniently defined by

$$M := \{(z^2, tz) \,|\, z \in \mathbb{C},\ t \in \mathbb{R},\ |z| = 1\}.$$

Figure 1.1: The Möbius strip



Identifying $\mathbb{C} = \mathbb{R}^2$ we have that

$$M \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4.$$

One can show that M is a non-orientable regular surface, see Problems 1.15 and 1.16.

1.7 The tangent bundle

The material in this section will not be used later in this text and may therefore be skipped on first reading.

Let $H : U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. We can collect the differentials dH_p for all $p \in U$ to form a single map

$$dH : U \times \mathbb{R}^m \to V \times \mathbb{R}^n, \quad (p, v) \mapsto (H(p), dH_p(v))$$

which is called the **global differential** of H. If $v = (v^1, \ldots, v^m)$ then

$$dH(p, v) = (H(p), \sum_{i=1}^{m} v^i \partial_i H(p)), \tag{1.2}$$

from which we see that dH is smooth.

We would like to generalize the notion of global differential to smooth maps between manifolds. To this end we introduce the **tangent bundle** of a manifold. If $M \subset \mathbb{R}^m$ is a k–manifold, the tangent bundle TM is defined by

$$TM := \{(p, v) \in M \times \mathbb{R}^m \,|\, v \in T_pM\}.$$

The map

$$\pi_M : TM \to M, \quad (p, v) \mapsto p$$

is called the **projection map** of the bundle. It is clearly smooth.

Lemma 1.7.1 *If V is an open subset of M then TV is an open subset of TM.*

Proof. For any $p \in V$ one has $T_pV = T_pM$. Therefore, $TV = (\pi_M)^{-1}V$. Since the projection map π_M is continuous, the tangent bundle TV is an open subset of TM. □

We will prove in a moment that the tangent bundle is a manifold, but before doing so, we extend the notation of global differential to smooth maps $F : M \to N$ between manifolds by

$$dF : TM \to TN, \quad dF(p, v) := (F(p), dF_p(v)).$$

The fact that dF is smooth follows immediately from the smoothness of the maps (1.2). We can now state the chain rule in a particularly simple form.

Theorem 1.7.1 (Chain rule, III) *If*

$$M \xrightarrow{F} N \xrightarrow{G} P$$

are smooth maps between manifolds then

$$d(G \circ F) = dG \circ dF.$$

Proof. This is just a reformulation of Theorem 1.4.3. □

Note that the global differential of the identity map $\mathrm{Id}_M : M \to M$ is the identity map $TM \to TM$. Therefore, if $F : M \to N$ is a diffeomorphism then $dF : TM \to TN$ is also a diffeomorphism.

Proposition 1.7.1 *If $M \subset \mathbb{R}^m$ is a k–manifold then the tangent bundle $TM \subset \mathbb{R}^m \times \mathbb{R}^m$ is a $2k$–manifold.*

Proof. If $p \in M$ then, because M is a manifold, there exist a diffeomorphism $F : U \to V$ from an open subset of $\mathbb{R}^k$ onto a neighbourhood of p in M. As observed above, the global differential $dF : TU \to TV$ is a diffeomorphism. Since $TU = U \times \mathbb{R}^k$, and TV is an open subset of M, the map dF is a local parametrization of TM. □

With the notion of tangent bundle at hand, we can give a new description of vector fields. To any vector field X on a manifold M we can associate the map

$$\bar{X} : M \to TM, \quad \bar{X}(p) = (p, X(p)).$$

Then we clearly have

$$\pi_M \circ \bar{X} = \mathrm{Id}_M. \tag{1.3}$$

Conversely, any map $\bar{X} : M \to TM$ satisfying (1.3) is given by a vector field on M. Moreover, X is smooth if and only if $\bar{X}$ is smooth.

The following proposition provides an application of the concept of tangent bundle.

Proposition 1.7.2 *Let $F : M \to N$ be a diffeomorphism and X a smooth vector field on M. Then there exists a unique smooth vector field Y on N such that, for all $p \in M$,*

$$dF_p(X_p) = Y_{F(p)}.$$

This vector field Y is called the **pushforward** of X by F and is usually denoted by F_*X.

Proof. Uniqueness is obvious. To prove existence, define

$$\bar{Y} := dF \circ \bar{X} \circ F^{-1}$$

and let Y be the corresponding vector field. $\square$

Problems

1.1 Prove that $\mathbb{R}^n - \{0\}$ is diffeomorphic to the product $S^{n-1} \times \mathbb{R}$ for every positive integer n.

1.2 Let $L \subset \mathbb{R}^2$ be the union of the non-negative parts of the coordinate axes, i.e.

$$L := ([0, \infty) \times \{0\}) \cup (\{0\} \times [0, \infty)).$$

Prove that L is not a manifold.

1.3 Let $M \subset \mathbb{R}^n$ be a non-empty k–manifold. Show that the dimension k is uniquely determined by M as a subset of $\mathbb{R}^n$.

1.4 Let $U \subset \mathbb{R}^m$ be an open set and $F : U \to \mathbb{R}^n$ a smooth map. Let $\Gamma_F \subset \mathbb{R}^m \times \mathbb{R}^n$ be the graph of F, i.e.

$$\Gamma_F := \{(u, F(u)) \,|\, u \in U\}.$$

Prove that Γ_F is an m–manifold.

1.5 Let $E \subset \mathbb{R}^n$ be a k–dimensional affine subspace, i.e. $E = p + V$ for some $p \in \mathbb{R}^n$ and k–dimensional linear subspace $V \subset \mathbb{R}^n$. Here,

$$p + V := \{p + v \,|\, v \in V\}.$$

Prove the following.

(a) If $v_1, \ldots, v_k$ is a basis for V then the map

$$\mathbb{R}^k \to E, \quad (x^1, \ldots, x^k) \mapsto p + \sum_{i=1}^{k} x^i v_i$$

is a diffeomorphism, hence E is a k–manifold.

(b) $T_q E = V$ for all $q \in E$.

1.6 For $i = 1, 2$ let $A_i \subset \mathbb{R}^{m_i}$ and $B_i \subset \mathbb{R}^{n_i}$ be arbitrary subsets, and let $F_i : A_i \to B_i$ be a smooth map. The product $A_1 \times A_2$ is a subset of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, which we identify with $\mathbb{R}^{m_1+m_2}$, and similarly for $B_1 \times B_2$. Show that the map

$$F_1 \times F_2 : A_1 \times A_2 \to B_1 \times B_2, \quad (a_1, a_2) \mapsto (F_1(a_1), F_2(a_2))$$

is smooth. Moreover, if both F_1 and F_2 are diffeomorphisms, then $F_1 \times F_2$ is also a diffeomorphism.

1.7 Use the result of Problem 1.6 to prove Proposition 1.2.1.

1.8 Given an integer $n \geq 2$, let

$$L := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x, y \text{ are linearly dependent}\}.$$

(a) Show that $L' := L - \{(0, 0)\}$ is a manifold and determine its dimension.

(b) Show that L is not a manifold.

1.9 Let $\Omega \subset \mathbb{R}^n$ be an open set and $g : \Omega \to \mathbb{R}^k$ a smooth map. Fix $p \in \Omega$ and let $k := \operatorname{rank}(dg_p)$ be the rank of dg_p (i.e. the dimension of its image). Show that there is a neighbourhood $W \subset \Omega$ of p such that for all $x \in W$ one has

$$\operatorname{rank}(dg_x) \geq \operatorname{rank}(dg_p).$$

1.10 (Lagrange multipliers) Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $g : \Omega \to \mathbb{R}^k$ be a smooth map with components $g^1, \ldots, g^k$. Let $S := g^{-1}(0)$. Prove the following.

(a) Given $p \in \Omega$, the gradients $\nabla g^1(p), \ldots, \nabla g^k(p)$ are linearly independent if and only if p is a regular point for g.

(b) Suppose $f : \Omega \to \mathbb{R}$ is a smooth function such that $f|_S$ has a local maximum or minimum at a point $p \in S$. If $\nabla g^1(p), \ldots, \nabla g^k(p)$ are linearly independent then there are real numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\nabla f(p) = \sum_{i=1}^{k} \lambda_i \nabla g^i(p).$$

Hint: Use the result of Problem 1.9 to show that there is a neighbourhood $W \subset \mathbb{R}^n$ such that $S \cap W$ is a manifold of dimension $n - k$. Observe that df_p vanishes on T_pS and deduce that df_p factors through dg_p, i.e. there is a linear map $A : \mathbb{R}^k \to \mathbb{R}$ such that $df_p = A \circ dg_p$.

1.11 Let $M \subset \mathbb{R}^n$ be a k–manifold. Suppose $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$ are continuous frames on an open set $V \subset M$. Let (a^i_j) be the $k \times k$ matrix of real-valued functions on V such that

$$Y_j = \sum_i a^i_j X_i.$$

Show that each function a^i_j is continuous, and a^i_j is smooth if both frames are smooth.

1.12 For $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$, where $n \geq 2$, the **generalized cross product** $v_1 \times \cdots \times v_{n-1}$ is the unique element of $\mathbb{R}^n$ such that for every $v_n \in \mathbb{R}^n$ one has

$$\det(v_1, \ldots, v_n) = \langle v_1 \times \cdots \times v_{n-1}, v_n \rangle,$$

where the left hand side is the determinant of the $n \times n$ matrix with columns $v_1, \ldots, v_n$. Verify the following properties of this product.

(a) $v_1 \times \cdots \times v_{n-1}$ is perpendicular to v_i for $i = 1, \ldots, n-1$.

(b) $v_1 \times \cdots \times v_{n-1} = 0$ if and only if $v_1, \ldots, v_{n-1}$ are linearly dependent.

(c) $\|v_1 \times \cdots \times v_{n-1}\| = 1$ if $\{v_i\}_{i=1,\ldots,n-1}$ is an orthonormal (indexed) set.

1.13 Extend the proof of Proposition 1.6.1 given for $n = 3$ to the general case $n \geq 1$. Moreover, explain why there is a one-to-one correspondence between the set of orientations of S and the set of smooth unit normal fields on S.

Hint: It may be convenient to use the generalized cross product from Problem 1.12.

1.14 Let H be the open right half-plane in the complex plane, i.e.

$$H := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}.$$

Let $\Omega \subset \mathbb{C}$ be the complement of the non-positive part of the real axis, i.e.

$$\Omega := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) \neq 0\}.$$

Show that the map

$$H \to \Omega, \quad z \mapsto z^2$$

is a diffeomorphism.

1.15 The aim of this problem is to show that the Möbius strip M, defined as

$$M := \{(z^2, tz) \,|\, z \in \mathbb{C},\ t \in \mathbb{R},\ |z| = 1\},$$

is a regular surface in $\mathbb{C}^2 = \mathbb{R}^4$.

(a) Let S^1_+ be the open right hemicircle in $S^1 \subset \mathbb{C}$, i.e.

$$S^1_+ := \{z \in S^1 \,|\, \mathrm{Re}(z) > 0\}.$$

Show that the map

$$F : S^1_+ \times \mathbb{R} \to M, \quad (z, t) \mapsto (z^2, tz)$$

is a local parametrization.

Hint: Let Ω be as in Problem 1.14. Use the result of that problem to construct a smooth map

$$G : \Omega \to \mathbb{C} \times \mathbb{R}$$

such that $G \circ F = \mathrm{Id}$.

(b) Given $q \in S^1 \subset \mathbb{C}$, show that the map

$$\mathbb{C}^2 \to \mathbb{C}^2 \quad (v, w) \mapsto (q^2 v, qw)$$

maps M diffeomorphically onto M. Combine this with (i) to conclude that M is a regular surface.

1.16 Prove that the Möbius strip, as defined in Example 1.6.2, is non-orientable.

1.17 Let $\phi : M \to M'$ be a local diffeomorphism between n–manifolds, and let Y' be smooth vector field on M'. Show that there is a unique smooth vector field $Y = \phi^*(Y')$ on M such that for all $p \in M$ one has

$$d\phi_p(Y_p) = Y'_{\phi(p)}.$$

1.18 Prove that the tangent bundle of the circle S^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Chapter 2

Lengths and distances

In this chapter we begin the study of the geometry of manifolds. Every tangent space of a manifold $M \subset \mathbb{R}^n$ inherits a scalar product from $\mathbb{R}^n$, and this allows us to define the notion of an isometry between two manifolds as well as the length of a smooth curve in M.

2.1 The first fundamental form

Let $M \subset \mathbb{R}^n$ be a manifold. For every point $p \in M$, the tangent space T_pM inherits a Euclidean scalar product g_p from the standard scalar product on $\mathbb{R}^n$, i.e.

$$g_p(v, w) = \langle v, w \rangle$$

for $v, w \in T_pM$. The rule that assigns to each $p \in M$ the scalar product g_p is called the **first fundamental form** of M. One often writes I_p instead of g_p.

Suppose $F : U \to M$ is a local parametrization and $(X_1, \ldots, X_n)$ the corresponding coordinate frame on $V := F(U)$. For every $p \in V$ the vectors $X_1(p), \ldots, X_n(p)$ form a basis for the tangent space T_pM. Therefore, the scalar product g_p is completely determined by the positive definite, symmetric matrix $(g_{ij}(p))$ given by

$$g_{ij}(p) := \langle X_i(p), X_j(p) \rangle.$$

(For a proof that the matrix is positive definite, see Problem 3.1.) In concrete calculations we will usually work with the corresponding functions $\tilde{g}_{ij} := g_{ij} \circ F$ on the parameter domain U. For $u \in U$ we have $X_i(F(u)) = \partial_i F(u)$, hence

$$\tilde{g}_{ij}(u) = g_{ij}(F(u)) = \langle \partial_i F(u), \partial_j F(u) \rangle.$$

From this we see that each function $\tilde{g}_{ij} : U \to \mathbb{R}$ is a smooth, hence $g_{ij} = \tilde{g}_{ij} \circ F^{-1} : V \to \mathbb{R}$ is smooth, too.

Example 2.1.1 Let E be a k–dimensional affine subspace of $\mathbb{R}^n$. Then E has a global parametrization of the form

$$F : \mathbb{R}^k \to E, \quad (x^1, \ldots, x^k) \mapsto p + \sum_{i=1}^{k} x^i v_i,$$

where $p, v_1, \ldots, v_k \in \mathbb{R}^n$ and $v_1, \ldots, v_k$ are linearly independent, see Problem 1.5. Then

$$\partial_i F \equiv v_i, \quad \tilde{g}_{ij} \equiv \langle v_i, v_j \rangle.$$

Example 2.1.2 Let S be the xy–plane in $\mathbb{R}^3$. Polar coordinates provide a local parametrization

$$F : (0, \infty) \times (0, 2\pi) \to S, \quad F(r, \theta) = (r \cos\theta, r \sin\theta, 0).$$

We calculate

$$\frac{\partial F}{\partial r} = (\cos\theta, \sin\theta, 0), \quad \frac{\partial F}{\partial \theta} = (-r \sin\theta, r \cos\theta, 0),$$

where we have written $\frac{\partial F}{\partial r}$ instead of $\frac{\partial F}{\partial r}(r, \theta)$ etc. for simplicity. This yields

$$\begin{aligned}
\tilde{g}_{11} &= \left\| \frac{\partial F}{\partial r} \right\|^2 = 1, \\
\tilde{g}_{12} = \tilde{g}_{21} &= \left\langle \frac{\partial F}{\partial r}, \frac{\partial F}{\partial \theta} \right\rangle = 0, \\
\tilde{g}_{22} &= \left\| \frac{\partial F}{\partial \theta} \right\|^2 = r^2.
\end{aligned}$$

Therefore,

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Example 2.1.3 We now consider the cylinder

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}.$$

This surface has a local parametrization

$$F : (0, 2\pi) \times \mathbb{R} \to S, \quad F(\theta, h) = (\cos\theta, \sin\theta, h).$$

In this case,

$$\frac{\partial F}{\partial \theta} = (-\sin\theta, \cos\theta, 0), \quad \frac{\partial F}{\partial h} = (0, 0, 1),$$

so that

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, for every $u \in U$ the differential dF_u maps the standard orthonormal basis for $\mathbb{R}^2$ onto an orthonormal basis for $T_{F(u)}S$, hence $dF_u : \mathbb{R}^2 \to T_{F(u)}S$ is a linear isometry.

Example 2.1.4 Spherical coordinates provide a local parametrization

$$F : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi) \to S^2, \quad F(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta).$$

We calculate

$$\begin{aligned} \frac{\partial F}{\partial \theta} &= (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta), \\ \frac{\partial F}{\partial \phi} &= (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0), \end{aligned}$$

from which we obtain

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}.$$

2.2 Isometries

A smooth map $\phi : M \to M'$ between manifolds of the same dimension is called a **local isometry** if for every $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}M'$ is a linear isometry, i.e. for all $v, w \in T_pM$ one has

$$\langle d\phi_p(v), d\phi_p(w)\rangle = \langle v, w\rangle.$$

This implies that ϕ is a local diffeomorphism, because a linear isometry is always injective and we assume that M, M' have the same dimension.

A bijective local isometry is called an **isometry** . Clearly, every isometry is a diffeomorphism.

Example 2.2.1 The local parametrization F in Example 2.1.3 is an isometry.

Example 2.2.2 We will construct a local isometry $\Omega \to S$, where S is the cone

$$S := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = z^2,\ z > 0\}$$

and $\Omega \subset \mathbb{R}^2$ is the complement of the non-negative part of the x–axis, i.e.

$$\Omega := \{(x, y) \in \mathbb{R}^2 \,|\, x < 0 \text{ or } y \neq 0\}.$$

As a stepping stone we use the open subset $U \subset \mathbb{R}^2$ defined by

$$U := \{(s, \theta) \in \mathbb{R}^2 \,|\, s > 0,\ 0 < \theta < 2\pi\}.$$

Fix a constant $a > 0$. We define a smooth map $F : U \to S$ by

$$F(s, \theta) := (s\cos(a\theta), s\sin(a\theta), s).$$

We calculate

$$\begin{aligned} \frac{\partial F}{\partial s} &= (\cos(a\theta), \sin(a\theta), 1), \\ \frac{\partial F}{\partial \theta} &= (-as\sin(a\theta), as\cos(a\theta), 0). \end{aligned}$$

This yields

$$\left\| \frac{\partial F}{\partial s} \right\|^2 = 2, \quad \left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial \theta} \right\rangle = 0, \quad \left\| \frac{\partial F}{\partial \theta} \right\|^2 = a^2 s^2.$$

Since the matrix $(\langle \partial_i F, \partial_j F \rangle)$ is positive definite at every point $(s, \theta) \in U$, the map F is a local diffeomorphism (see Problem3.1). On the other hand, for the map $\phi : U \to \Omega$ given by

$$\phi(s, \theta) := (as\cos\theta, as\sin\theta)$$

we find

$$\frac{\partial \phi}{\partial s} = (a\cos\theta, a\sin\theta), \quad \frac{\partial \phi}{\partial \theta} = (-as\sin\theta, as\cos\theta)$$

and

$$\left\| \frac{\partial \phi}{\partial s} \right\|^2 = a^2, \quad \left\langle \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial \theta} \right\rangle = 0, \quad \left\| \frac{\partial \phi}{\partial \theta} \right\|^2 = a^2 s^2.$$

For $a = \sqrt{2}$ we therefore have

$$\langle \partial_i F, \partial_j F \rangle = \langle \partial_i \phi, \partial_j \phi \rangle$$

at every point $(s, \theta) \in U$, hence $F \circ \phi^{-1} : \Omega \to S$ is a local isometry.

2.3 Parametrized curves

By a **parametrized curve** in a manifold M we mean a smooth map $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an interval. Such a curve γ is called **regular** if $\gamma'(t) \neq 0$ for all $t \in I$. If J is another interval and $u : J \to I$ a diffeomorphism then the curve $\gamma \circ u : J \to M$ is called a **reparametrization** of γ. We say the reparametrization is **forward** if u is increasing and **backward** if u is decreasing. The **speed** of γ at a point $t \in I$ is the scalar $\|\gamma'(t)\|$. If $\|\gamma'(t)\| = 1$ for all t then we say γ has **unit speed** .

Lemma 2.3.1 *Every regular curve $\gamma : I \to M$ has a unit speed forward reparametrization*

Proof. Choose $s_0 \in I$ and define $f : I \to \mathbb{R}$ by

$$f(s) := \int_{s_0}^{s} \|\gamma'(t)\| dt.$$

Then

$$f'(s) = \|\gamma'(s)\| > 0,$$

so f is strictly increasing, and the image $J := f(I)$ is an interval. The inverse function

$$u := f^{-1} : J \to I$$

satisfies

$$u'(t) = \frac{1}{f'(u(t))} = \frac{1}{\|\gamma'(u(t))\|}.$$

By the chain rule,

$$(\gamma \circ u)'(t) = u'(t) \cdot \gamma'(u(t)) = \frac{\gamma'(u(t))}{\|\gamma'(u(t))\|}.$$

Hence, the reparametrized curve $\gamma \circ u$ has unit speed. $\square$

The **length** of a smooth curve $\gamma : [a, b] \to M$ is defined by

$$L[\gamma] := \int_a^b \|\gamma'(t)\| dt.$$

Lemma 2.3.2 *Every reparametrization of γ has the same length as γ.*

Proof. Let $u : [a', b'] \to [a, b]$ be a diffeomorphism. Then either $u'(t)$ is either always positive or always negative, so

$$u' = \epsilon |u'|$$

for some constant $\epsilon = \pm 1$. By the chain rule,

$$(\gamma \circ u)'(t) = u'(t)\gamma'(u(t)),$$

so we get

$$\begin{aligned} L[\gamma \circ u] &= \int_{a'}^{b'} |u'(t)| \|\gamma'(u(t))\| dt \\ &= \epsilon \int_{a'}^{b'} u'(t) \|\gamma'(u(t))\| dt \\ &= \epsilon \int_{u(a')}^{u(b')} \|\gamma'(s)\| ds \\ &= \int_{a}^{b} \|\gamma'(s)\| ds \\ &= L[\gamma]. \qquad \square \end{aligned}$$

Problems

2.1 Show that the map

$$\phi : \mathbb{R}^2 \to S^1 \times S^1, \quad (s, t) \mapsto ((\cos t, \sin t), (\cos t, \sin t))$$

is a local isometry.

2.2 Let $\phi : M \to M'$ be a smooth map between manifolds of the same dimension. Show that ϕ is a local isometry if and only if ϕ preserves the length of curves in the following sense: For every smooth curve $\gamma : [a, b] \to M$ one has

$$L[\phi \circ \gamma] = L[\gamma].$$

Chapter 3

Classical surface theory

In 1827, Carl Friedrich Gauss published the paper [3] where he introduced what is now called 'Gauss curvature', a real-valued function defined on any regular surface in $\mathbb{R}^3$. His *Theorema Egregium* (remarkable theorem) states that Gauss curvature is preserved by local isometries.

In this chapter we define Gauss curvature and study it for two special classes of surfaces, namely ruled surfaces and surfaces of revolution. A modern proof of the Theorema Egregium will be given in Chapters 4 and 5.

This chapter owes much to the exposition in [1].

3.1 Gauss curvature

Let $S \subset \mathbb{R}^3$ be a regular surface. The **Gauss curvature** $K : S \to \mathbb{R}$ is defined as follows. Given $p \in S$, choose a smooth unit normal field N defined in a neighbourhood W of p in S. We now look at the differential of N as a map $W \to S^2$. Because

$$T_{N(p)}S^2 = N(p)^\perp = T_pS,$$

the differential

$$dN_p : T_pS \to T_{N(p)}S^2$$

is in fact an endomorphism of T_pS. The Gauss curvature at p is defined to be the determinant of this endomorphism, i.e.

$$K(p) = \det(dN_p).$$

Then $K(p)$ is independent of the choice of N, because

$$\det(d(-N)_p) = \det(-dN_p) = \det(dN_p).$$

The linear map

$$W_p = -dN_p : T_pS \to T_pS$$

is called the **Weingarten map**. Clearly,

$$\det(W_p) = \det(dN_p) = K(p).$$

The bilinear map

$$I\!I_p : T_pS \times T_pS \to \mathbb{R}, \quad (u, v) \mapsto \langle W_p(u), v\rangle$$

is called the **second fundamental form** relative to the normal field N.

Our next goal is to describe the second fundamental form and the Gauss curvature in terms of a local parametrization $F : U \to S$, where $U \subset \mathbb{R}^2$ is an open set. Let X_1, X_2 be the corresponding coordinate vector fields on $V := F(U)$, and let N be a smooth unit normal field on V. For $p \in V$, the second fundamental form $I\!I_p$ is completely determined by the 2×2 matrix $(h_{ij}(p))$ given by

$$h_{ij}(p) := I\!I_p(X_i(p), X_j(p)).$$

In calculations we will often work with the corresponding functions $\tilde{h}_{ij} := h_{ij} \circ F$ on U. For $u \in U$, $p = F(u)$ we find

$$\tilde{h}_{ij}(u) = I\!I_p(\partial_i F(u), \partial_j F(u)) = \langle W_p(\partial_i F(u)), \partial_j F(u)\rangle.$$

Let

$$\tilde{N} := N \circ F : U \to \mathbb{R}^3.$$

Proposition 3.1.1 $\tilde{h}_{ij} = \langle \partial_i \partial_j F, \tilde{N}\rangle$.

Proof. For any $u \in U$, the partial derivative $\partial_j F(u)$ lies in the tangent space $T_{F(u)}S$ whereas $\tilde{N}(u)$ is perpendicular to it, hence $\langle \partial_j F, \tilde{N}\rangle = 0$. Differentiating this equality we get

$$0 = \partial_i \langle \partial_j F, \tilde{N}\rangle = \langle \partial_i \partial_j F, \tilde{N}\rangle + \langle \partial_j F, \partial_i \tilde{N}\rangle.$$

Let $p := F(u)$. The chain rule gives

$$(\partial_i \tilde{N})(u) = \partial_i (N \circ F)(u) = dN_p(\partial_i F(u)) = -W_p(\partial_i F(u)).$$

Putting this together we get

$$\langle \partial_i \partial_j F(u), \tilde{N}(u)\rangle = \langle \partial_j F(u), W_p(\partial_i F(u))\rangle = \tilde{h}_{ij}(u). \qquad \square$$

Since $\partial_1 \partial_2 F = \partial_2 \partial_1 F$, the proposition shows that $\tilde{h}_{12} = \tilde{h}_{21}$, and similarly

$$h_{12} = h_{21}.$$

Corollary 3.1.1 *For every $p \in S$, the Weingarten map $W_p : T_pS \to T_pS$ is self-adjoint, i.e. for all $v, w \in T_pS$ one has*

$$\langle W_p(v), w\rangle = \langle v, W_p(w)\rangle.$$

Proof. After choosing a local parametrization of S around p, the corollary follows because $h_{12} = h_{21}$ and $X_1(p), X_2(p)$ form a basis for T_pS. □

The eigenvalues κ_1, κ_2 of W_p are called the **principal curvatures** of S at p. We observe that

$$K(p) = \det(W_p) = \kappa_1\kappa_2.$$

The components of the first and second fundamental forms make up two symmetric 2×2 matrices $G = (g_{ij})$ and $H = (h_{ij})$. We now express the Gauss curvature K of S in terms of the determinants of these matrices.

Theorem 3.1.1 $K = \dfrac{\det(H)}{\det(G)}$.

More precisely, this equation holds on $V = F(U)$, where $F : U \to S$ is the local parametrization used to define G and H.

Proof. Given $p \in V$, let $e_i = X_i(p)$. Then (e_1, e_2) is a basis for T_pS, and

$$g_{ij}(p) = \langle e_i, e_j\rangle.$$

Let $A = (a_{ij})$ be the matrix of the Weingarten map W_p with respect to this basis, so that

$$W_p e_j = \sum_i a_{ij} e_i.$$

Then

$$h_{ij}(p) = \langle W_p e_i, e_j\rangle = \langle \sum_k a_{ki} e_k, e_j\rangle = \sum_k g_{jk}(p) a_{ki}.$$

Writing G instead of $G(p)$ and similarly for H, we recognize the last sum as the (ji) entry of the matrix product GA. This means that the transpose of H is

$$H^T = GA,$$

hence

$$\det(H) = \det(H^T) = \det(G)\det(A).$$

Recalling that $K(p) = \det(A)$ and $\det(G) \neq 0$, this proves the theorem. □

In calculations one often works with $\tilde{K} := K \circ F : U \to \mathbb{R}$. In terms of the matrices

$$\tilde{G} := (\tilde{g}_{ij}), \quad \tilde{H} := (\tilde{h}_{ij})$$

the theorem can be rephrased as

$$\tilde{K} = \frac{\det(\tilde{H})}{\det(\tilde{G})}. \tag{3.1}$$

Since one often chooses

$$\tilde{N} = \frac{\partial_1 F \times \partial_2 F}{\|\partial_1 F \times \partial_2 F\|},$$

the following lemma may be useful.

Lemma 3.1.1 $\det(\tilde{G}) = \|\partial_1 F \times \partial_2 F\|^2$.

Proof. The cross product on $\mathbb{R}^3$ satisfies

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$$

for all $u, v \in \mathbb{R}^3$. This yields

$$\begin{aligned} \|\partial_1 F \times \partial_2 F\|^2 &= \|\partial_1 F\|^2\|\partial_2 F\|^2 - \langle \partial_1 F, \partial_2 F\rangle^2 \\ &= \tilde{g}_{11}\tilde{g}_{22} - (\tilde{g}_{12})^2 \\ &= \det(\tilde{G}). \qquad \square \end{aligned}$$

3.2 Some geometric properties of Gauss curvature

Proposition 3.2.1 *Let $S \subset \mathbb{R}^3$ be a regular surface. Suppose $p \in S$ and $r > 0$ is a constant such that*

- $\|x\| \leq r$ *for all* $x \in S$,
- $\|p\| = r$.

Then

$$K(p) \geq \frac{1}{r^2}.$$

Proof. Let N be a smooth unit normal field defined in a neighbourhood W of p in S. Let $v \in T_pS$ and choose a smooth curve $\gamma : (-\epsilon, \epsilon) \to W$ such that

$$\gamma(0) = p, \quad \gamma'(0) = v.$$

We consider the function

$$f(t) = \frac{1}{2}\|\gamma(t)\|^2.$$

The first two derivatives are

$$\begin{aligned} f'(t) &= \langle \gamma'(t), \gamma(t)\rangle, \\ f''(t) &= \langle \gamma''(t), \gamma(t)\rangle + \|\gamma'(t)\|^2. \end{aligned}$$

Because f has a maximum at $t = 0$, we have

(i) $0 = f'(0) = \langle v, p\rangle$,

(ii) $0 \geq f''(0) = \langle \gamma''(0), p\rangle + \|v\|^2$.

Since (i) holds for all $v \in T_pS$, we have $p \perp T_pS$, so we may assume that $N(p) = -p/r$. Now observe that

$$0 = \langle \gamma'(t), N(\gamma(t))\rangle$$

for all t, so

$$0 = \frac{d}{dt}\langle \gamma'(t), N(\gamma(t))\rangle = \langle \gamma''(t), N(\gamma(t))\rangle + \langle \gamma'(t), dN_{\gamma(t)}(\gamma'(t))\rangle.$$

For $t = 0$ we get

$$\langle v, W_p(v)\rangle = \langle \gamma''(0), N(p)\rangle = -\frac{1}{r}\langle \gamma''(0), p\rangle \geq \frac{1}{r}\|v\|^2,$$

where the inequality follows from (ii) above. If v is in fact an eigenvector of W_p, say $W_p(v) = \kappa v$, then

$$\kappa\|v\|^2 = \langle v, W_p(v) \geq \frac{1}{r}\|v\|^2,$$

so $\kappa \geq 1/r$. Now let κ_1, κ_2 be the eigenvalues of W_p. Then

$$K(p) = \kappa_1\kappa_2 \geq \frac{1}{r^2}. \qquad \square$$

Corollary 3.2.1 *If $S \subset \mathbb{R}^3$ is a compact, non-empty regular surface then there is a point $p \in S$ such that $K(p) > 0$.*

Proof. Let p be a point on S where the function

$$S \to \mathbb{R}, \quad x \mapsto \|x\|^2$$

has a maximum, and let $r = \|p\|$. Since S is a surface, it cannot consist of the origin alone, hence $r > 0$. Therefore,

$$K(p) \geq \frac{1}{r^2} > 0. \qquad \square$$

The following theorem describes a surface locally as the graph of a function.

Theorem 3.2.1 *Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$, and ξ_1, ξ_2, ξ_3 an orthonormal basis for $\mathbb{R}^3$ such that $\xi_1, \xi_2 \in T_pS$.*

(i) *There exists a local parametrization of S around p of the form*

$$F(u^1, u^2) = p + u^1\xi_1 + u^2\xi_2 + f(u^1, u^2)\xi_3,$$

where $f : U \to \mathbb{R}$ is a smooth function satisfying

$$f(0,0) = 0; \quad \partial_i f(0,0) = 0 \text{ for } i = 1, 2.$$

(ii) *If F is any local parametrization as in (i) then the Gauss curvature of S at p agrees with the determinant of the Hessian matrix of f at the origin, i.e.*

$$K(p) = \det(\partial_i \partial_j f(0,0)).$$

Proof. (i) Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by

$$\pi\left(\sum_{i=1}^{3} a^i \xi_i\right) := (a^1, a^2)$$

for $a^i \in \mathbb{R}$. Let

$$\phi : S \to \mathbb{R}^2, \quad x \mapsto \pi(x - p).$$

At any point $x \in S$ the differential of ϕ is the restriction of π, i.e.

$$d\phi_x(v) = \pi(v)$$

for $v \in T_xS$. Therefore, $d\phi_p$ maps the basis ξ_1, ξ_2 for T_pS to the basis $(1,0), (0,1)$ for $\mathbb{R}^2$, so $d\phi_p : T_pS \to \mathbb{R}^2$ is an isomorphism. By the inverse

function theorem, Theorem 1.4.1, ϕ maps some neighbourhood V of p in S diffeomorphically onto a neighbourhood U of $(0,0)$ in $\mathbb{R}^2$. Let

$$F := \phi^{-1} : U \to V.$$

For any point $(u^1, u^2) \in U$ we can express $F(u^1, u^2) - p$ in terms of the basis ξ_1, ξ_2, ξ_3 for $\mathbb{R}^3$. This gives rise to real-valued functions f^1, f^2, f^3 on U such that

$$F(u^1, u^2) - p = \sum_{i=1}^{3} f^i(u^1, u^2)\xi_i.$$

Applying π to both sides of this equation we get

$$f^i(u^1, u^2) = u^i, \quad i = 1, 2.$$

Writing $f := f^3$ we then have

$$F(u^1, u^2) = p + u^1\xi_1 + u^2\xi_2 + f(u^1, u^2)\xi_3.$$

Since $F(0,0) = p$ we have $f(0,0) = 0$. The partial derivatives of F are

$$\partial_i F = \xi_i + \partial_i f \cdot \xi_3.$$

Because the vectors ξ_1, ξ_2 and $\partial_i F(0,0)$ lie in the tangent space T_pS whereas ξ_3 does not, we must have $\partial_i f(0,0) = 0$.

(ii) Let $\tilde{G} = (\tilde{g}_{ij})$ be the matrix of the first fundamental form. Since $\partial_i F(0,0) = \xi_i$ we see that $\tilde{G}$ is the identity matrix. Choose a smooth unit normal field N defined in some neighbourhood of p in S such that $N(p) = \xi_3$, and let $\tilde{H} = (\tilde{h}_{ij})$ be the matrix of the second fundamental form relative to N. The second order partial derivatives of F are

$$\partial_i \partial_j F = \partial_i \partial_j f \cdot \xi_3,$$

hence

$$\tilde{h}_{ij}(0,0) = \langle \partial_i \partial_j F(0,0), N(p) \rangle = \partial_i \partial_j f(0,0).$$

Thus, $\tilde{H}(0,0)$ is the Hessian matrix of f at the origin, so

$$K(p) = \frac{\det(\tilde{H}(0,0))}{\det(\tilde{G}(0,0))} = \det(\partial_i \partial_j f(0,0)). \qquad \square$$

The theorem provides a geometric interpretation of Gauss curvature, at least at points where it is non-zero. If S is a regular surface, a point $p \in S$ is called **elliptic** if $K(p) > 0$ and **hyperbolic** if $K(p) < 0$. Near p, the

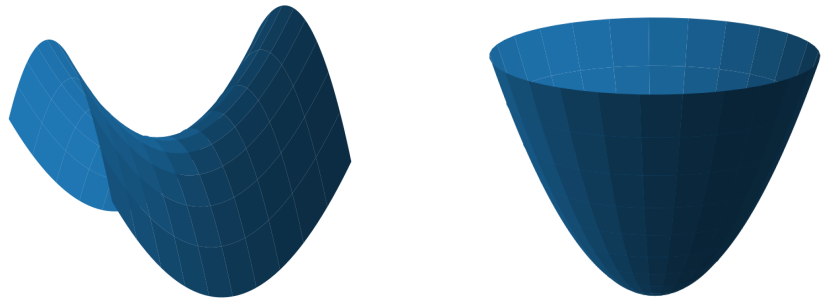


Figure 3.1: Surfaces with Gauss curvature $K < 0$ (left) and $K > 0$ (right).

surface can be approximated by an elliptic paraboloid if $K(p) > 0$ and an hyperbolic paraboloid (or saddle surface) if $K(p) < 0$.

If E is any affine plane in $\mathbb{R}^3$ then $\mathbb{R}^3 - E$ has two connected components. A subset $A \subset \mathbb{R}^3$ is said to lie **completely on one side of** E if A is contained in one of the connected components of $\mathbb{R}^3 - E$. From the last theorem we obtain the following corollary.

Corollary 3.2.2 **(i)** *If $K(p) > 0$ then p has a neighbourhood W in S such that $W - \{p\}$ lies completely on one side of the affine tangent plane $p + T_pS$.*

(ii) *If $K(p) < 0$ then any neighbourhood of p in S contains points from both sides of $p + T_pS$.* $\square$

3.3 Ruled surfaces

A regular surface $S \subset \mathbb{R}^3$ is called a **ruled surface** if, around every point in S, there is a local parametrization of the form

$$F : I \times J \to S, \quad (t, s) \mapsto c(t) + sv(t),$$

where I, J are open intervals and

$$c : I \to \mathbb{R}^3, \quad v : I \to \mathbb{R}^3$$

are smooth functions such that $v(t) \neq 0$ for all t.

Example 3.3.1 If $C \subset \mathbb{R}^2$ is a regular curve then the product $S := C \times \mathbb{R} \subset \mathbb{R}^3$ is a regular surface called a **generalized cylinder**. If $\gamma : I \to C$ is a

local parametrization then

$$F : I \times \mathbb{R} \to S, \quad (t, s) \mapsto (\gamma^1(t), \gamma^2(t), s)$$

is a local parametrization of S. In this case, one can take

$$c(t) = (\gamma^1(t), \gamma^2(t), 0), \quad v(t) = (0, 0, 1).$$

Example 3.3.2 Let $C \subset \mathbb{R}^3$ be a regular curve contained in an affine plane E, and let $p \in \mathbb{R}^3 - E$. Then

$$S := \{(1 - s)a + sp \,|\, s < 1,\ a \in C\}$$

is a regular surface called a **generalized cone**. If $c : I \to C$ is a local parametrization then

$$F : I \times (-\infty, 1) \to S, \quad (t, s) \mapsto (1 - s)c(t) + sp$$

is a local parametrization of S (see Problem 3.3). Rewriting $F(t, s)$ as

$$F(t, s) = c(t) + s \cdot (p - c(t))$$

and taking $v(t) = p - c(t)$ we see that S is a ruled surface.

Example 3.3.3 The hyperbolic paraboloid

$$S := \{(x, y, z) \in \mathbb{R}^3 \,|\, z = xy\}$$

has a global parametrization

$$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3, \quad (t, s) \mapsto (t, s, ts).$$

Rewriting

$$F(t, s) = (t, 0, 0) + s(0, 1, t)$$

we conclude that S is a ruled surface.

Proposition 3.3.1 *Any ruled surface $S \subset \mathbb{R}^3$ has Gauss curvature $K \leq 0$.*

Proof. We will apply Theorem 3.1.1 to a local parametrization of S of the form

$$F(t, s) = c(t) + sv(t).$$

We observe that $\partial_2\partial_2 F = 0$, so by Proposition 3.1.1 we have $h_{22} = 0$. Therefore,

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & 0 \end{pmatrix},$$

hence $\det H = -h_{12}^2 \leq 0$. Because the matrix $G = (g_{ij})$ has positive determinant, Theorem 3.1.1 yields

$$K = \frac{\det(H)}{\det(G)} \leq 0. \qquad \square$$

3.4 Surfaces of revolution

Let $C \subset \mathbb{R}^2$ be a regular curve contained in the right half-plane, i.e. every point in C has the form (x, z) with $x > 0$. If we regard C as a curve in the xz–plane in $\mathbb{R}^3$ and rotate it around the z–axis, it sweeps out a **surface of revolution**. A precise definition is

$$S := \{(x\cos\theta, x\sin\theta, z) \,|\, (x, z) \in C,\ \theta \in \mathbb{R}\}.$$

Lemma 3.4.1 *If S is a surface of revolution as above then the map*

$$f : C \times S^1 \mapsto S, \quad ((x, z), (a, b)) \mapsto (ax, bx, z)$$

is a diffeomorphism. Therefore, S is a regular surface.

Proof. The map f is clearly bijective, and its inverse

$$f^{-1}(u, v, w) = \left(\left(\sqrt{u^2 + v^2}, w\right), \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}\right)\right)$$

is also smooth. $\square$

For $(p, z) \in C \times S^1$, the curve $f(\{p\} \times S^1)$ is called a **parallel** of S, and the curve $f(C \times \{z\})$ is called a **meridian** of S.

Example 3.4.1 For $0 < b < a$ we consider the circle

$$C := \{(x, z) \,|\, (x - a)^2 + z^2 = b^2\}.$$

The corresponding surface of revolution is a torus, diffeomorphic to $S^1 \times S^1$.

Let again S be an arbitrary surface of revolution and $\gamma : I \to C$ a local parametrization, where I is an open interval. Let r, h be the components of

γ, so that $\gamma(t) = (r(t), h(t))$. It follows from Lemma 3.4.1 that for any open interval J of length at most 2π the map

$$F : I \times J \to S, \quad (t, \theta) \mapsto (r(t)\cos\theta, r(t)\sin\theta, h(t)) \tag{3.2}$$

is a local parametrization. We will now use this to calculate the Gauss curvature of S. We find

$$\begin{aligned} \frac{\partial F}{\partial t} &= (\dot{r}(t)\cos\theta, \dot{r}(t)\sin\theta, \dot{h}(t)), \\ \frac{\partial F}{\partial \theta} &= (-r(t)\sin\theta, r(t)\cos\theta, 0), \end{aligned} \tag{3.3}$$

which yields

$$\tilde{g}_{11} = \dot{r}(t)^2 + \dot{h}(t)^2, \quad \tilde{g}_{12} = 0, \quad \tilde{g}_{22} = r(t)^2. \tag{3.4}$$

Furthermore,

$$\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta} = (-r(t)\dot{h}(t)\cos\theta, -r(t)\dot{h}(t)\sin\theta, r(t)\dot{r}(t)),$$

so

$$\det(\tilde{G}) = \left\| \frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta} \right\|^2 = r(t)^2(\dot{r}(t)^2 + \dot{h}(t)^2).$$

We now take

$$\tilde{N} = \frac{\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta}}{\|\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial \theta}\|} = \frac{(-\dot{h}(t)\cos\theta, -\dot{h}(t)\sin\theta, \dot{r}(t))}{\sqrt{\dot{r}(t)^2 + \dot{h}(t)^2}}.$$

In order to compute the components $\tilde{h}_{ij}$ of the second fundemental form we first calculate the second order partial derivatives of F.

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} &= (\ddot{r}(t)\cos\theta, \ddot{r}(t)\sin\theta, \ddot{h}(t)), \\ \frac{\partial^2 F}{\partial t \partial \theta} &= (-\dot{r}(t)\sin\theta, \dot{r}(t)\cos\theta, 0,), \\ \frac{\partial^2 F}{\partial \theta^2} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0). \end{aligned}$$

The components $\tilde{h}_{ij}$ turn out to be independent of θ, so we drop t, θ from notation. We get

$$\begin{aligned}
\tilde{h}_{11} &= \frac{-\ddot{r}\dot{h} + \dot{r}\ddot{h}}{\sqrt{\dot{r}^2 + \dot{h}^2}}, \\
\tilde{h}_{12} &= 0, \\
\tilde{h}_{22} &= \frac{r\dot{h}}{\sqrt{\dot{r}^2 + \dot{h}^2}},
\end{aligned}$$

and

$$\det(\tilde{H}) = \frac{r\dot{h}(\dot{r}\ddot{h} - \ddot{r}\dot{h})}{\dot{r}^2 + \dot{h}^2}.$$

Combining this with (3.1) we obtain

$$\tilde{K} = \frac{\det(\tilde{H})}{\det(\tilde{G})} = \frac{\dot{h}(\dot{r}\ddot{h} - \ddot{r}\dot{h})}{r(\dot{r}^2 + \dot{h}^2)^2}. \tag{3.5}$$

This formula simplifies a lot if we assume that the parametrized curve γ has unit speed.

Proposition 3.4.1 *Let C be a regular curve in the open right half-plane and S the corresponding surface of revolution. Let $\gamma(t) = (r(t), h(t))$ be a local parametrization of C. If γ has unit speed, i.e. if $\dot{r}^2 + \dot{h}^2 = 1$, then for any local parametrization F of S as in (3.2) the Gauss curvature of S satisfies*

$$\tilde{K}(t, \theta) = -\frac{\ddot{r}(t)}{r(t)}.$$

Proof. Differentiating the equation $\dot{r}^2 + \dot{h}^2 = 1$ we get $2(\dot{r}\ddot{r} + \dot{h}\ddot{h}) = 0$. Substituting for $\dot{h}\ddot{h}$ in (3.5) we find that

$$\tilde{K} = \frac{\dot{r}(\dot{h}\ddot{h}) - \ddot{r}\dot{h}^2}{r} = \frac{\dot{r}(-\dot{r}\ddot{r}) - \ddot{r}\dot{h}^2}{r} = -\frac{\ddot{r}}{r}. \qquad \square$$

The last proposition can be used to classify surfaces of revolution with constant Gauss curvature. We will carry out this classification for $K < 0$, leaving the analogous cases $K = 0$ and $K > 0$ to the reader, see Problem 3.8. We prove the result in two parts: First existence, then uniqueness.

Proposition 3.4.2 *Let c, ω be positive real numbers. If C is the plane curve parametrized by one of the functions $\gamma(t) = (r(t), h(t))$ listed in (I)-(III) below, then the corresponding surface of revolution S has constant Gauss curvature $K = -\omega^2$.*

(I) $r(t) = c\sinh(\omega t), \quad h(t) = \int_0^t \sqrt{1 - c^2\omega^2\cosh^2(\omega x)}dx,$

(II) $r(t) = c\cosh(\omega t), \quad h(t) = \int_0^t \sqrt{1 - c^2\omega^2\sinh^2(\omega x)}dx,$

(III) $r(t) = e^{\omega t}, \quad h(t) = \int_{t_0}^t \sqrt{1 - \omega^2 e^{2\omega x}}dx,$ *where* $t_0 = -\frac{\log\omega}{\omega}$.

In (I) the functions are defined for all $t > 0$ *for which* $c\omega\cosh(\omega t) < 1$, *whereas in (II) they are defined when* $-1 < c\omega\sinh(\omega t) < 1$. *In (III) the functions are defined for* $t < t_0$.

Proof. In each case it is easily verified that $\dot{r}^2 + \dot{h}^2 = 1$ and $\ddot{r} = \omega^2 r$. □

The surface of revolution obtained from the curve in case (III) is called the **pseudosphere**.

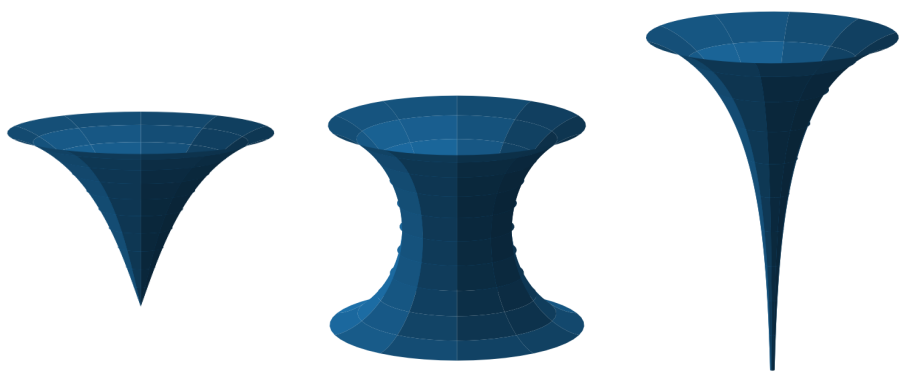


Figure 3.2: The surfaces of revolution described in Theorem 3.4.2 (I)-(III), where $c = 1/4$ in the leftmost surface, and $c = 1/2$ in the middle one. The rightmost surface is the pseudosphere.

Proposition 3.4.3 *Let* C *be a connected regular curve in the open right half-plane such that the corresponding surface of rotation* S *has constant Gauss curvature* $K < 0$. *Then there is a transformation* $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ *of the form*

$$\psi(x, y, z) = (x, y, \epsilon z + z_0),$$

where $\epsilon = \pm 1$ *and* $z_0 \in \mathbb{R}$, *such that* $\psi(S)$ *is an open subset of one of the surfaces of revolution obtained in Proposition 3.4.2.*

Proof. Suppose $K = -\omega^2$, where $\omega > 0$. By Corollary 3.2.1, S cannot be compact, hence C is not compact. One can show that every connected 1–manifold is diffeomorphic to either S^1 or $\mathbb{R}$. It follows that C is diffeomorphic to $\mathbb{R}$, so by Lemma 2.3.1 it admits a global parametrization $\gamma : I \to C$ of unit speed for some open interval I. Let $\gamma(t) = (r(t), h(t))$ as before. By Proposition 3.4.1 we must have

$$\ddot{r}(t) = \omega^2 r(t).$$

From the theory of ordinary differential equations we know that there exist real numbers a, b such that

$$r(t) = a\cosh(\omega t) + b\sinh(\omega t).$$

We now distinguish between three cases.

Case 1: $a^2 < b^2$. Let

$$c := b\sqrt{1 - \frac{a^2}{b^2}}.$$

Then $c^2 = b^2 - a^2$. Let t_1 be the unique real number such that

$$\sinh(\omega t_1) = \frac{a}{c}.$$

Because $\cosh x > 0$ and $\cosh^2 x - \sinh^2 x = 1$ for all real numbers x, we have

$$\cosh(\omega t_1) = \frac{b}{c}.$$

Applying the addition formula for $\sinh x$ we find that

$$r(t) = c\sinh(\omega(t + t_1)).$$

We may clearly assume $t_1 = 0$. Then

$$r(t) = c\sinh(\omega t), \quad \dot{r}(t) = c\omega\cosh(\omega t).$$

We may also assume $c > 0$ (otherwise we consider instead the reparametrized curve $\gamma(-t)$). Since we require $r(t) > 0$, we need $t > 0$. By assumption, $\dot{r}^2 + \dot{h}^2 = 1$, hence

$$\dot{h}(t) = \epsilon\sqrt{1 - c^2\omega^2\cosh^2(\omega t)}$$

for some constant $\epsilon = \pm 1$, and

$$h(t) = h_0 + \epsilon\int_0^t \sqrt{1 - c^2\omega^2\cosh^2(\omega x)}dx$$

for some real number h_0.

Case 2: $a^2 > b^2$. This is analogous to Case 1. Let

$$c := a\sqrt{1 - \frac{b^2}{a^2}}.$$

Then $c^2 = a^2 - b^2$. Let t_1 be the unique real number such that

$$\sinh(\omega t_1) = \frac{b}{c}.$$

Then $\cosh(\omega t_1) = a/c$. From the addition formula for cosh we see that

$$r(t) = c\cosh(\omega(t + t_1)).$$

We require $r > 0$, so $c > 0$. We may assume that $t_1 = 0$, so that

$$r(t) = c\cosh(\omega t), \quad \dot{r}(t) = c\omega\sinh(\omega t).$$

As in Case 1, this yields

$$h(t) = h_0 + \epsilon\int_0^t \sqrt{1 - c^2\omega^2\sinh^2(\omega x)}dx$$

for some $\epsilon = \pm 1$ and $h_0 \in \mathbb{R}$.

Case 3: $a^2 = b^2$, or equivalently, $a = \pm b$. After perhaps replacing $\gamma(t)$ with $\gamma(-t)$, we can arrange that $a = b$. Then

$$r(t) = a(\cosh(\omega t) + \sinh(\omega t)) = ae^{\omega t}.$$

Since $r > 0$, we must have $a > 0$. After replacing $\gamma(t)$ by $\gamma(t - \omega^{-1}\log a)$ we have

$$r(t) = e^{\omega t}, \quad \dot{r}(t) = \omega e^{\omega t}.$$

Then

$$\dot{h}(t) = \epsilon\sqrt{1 - \omega^2 e^{2\omega t}}$$

for some constant $\epsilon = \pm 1$. For the formula for $\dot{h}(t)$ to be well-defined, we need $t \leq t_0$, where $t_0 = -\frac{\log\omega}{\omega}$. In addition, γ is to be defined on an open interval, so we actually need $t < t_0$. For such t we have

$$h(t) = h_0 + \epsilon\int_{t_0}^t \sqrt{1 - \omega^2 e^{2\omega x}}dx,$$

for some $h_0 \in \mathbb{R}$. $\square$

Problems

3.1 Let M be a manifold and $U \subset \mathbb{R}^k$ an open subset. Let $F : U \to M$ be a smooth map. For a fixed $x \in U$ let $A = (a_{ij})$ be the symmetric matrix

$$a_{ij} := \langle \partial_i F(x), \partial_j F(x) \rangle.$$

Prove the following.

(a) For all $v, w \in \mathbb{R}^k$,

$$\langle v, Aw \rangle = \langle dF_x(v), dF_x(w) \rangle.$$

(b) All eigenvalues of A are non-negative.

(c) A is positive definite (i.e. all eigenvalues are positive) if and only if the differential dF_x is injective.

3.2 Let

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 1 \text{ and } x, y, z > 0\}.$$

(a) Show that S is a regular surface.

(b) Calculate the Gauss curvature of S.

3.3 Verify that every generalized cone as in Example 3.3.2 is a regular surface.

3.4 Let $F(t, s) = c(t) + sv(t)$ be a local parametrization of a ruled surface. Show that

$$K(F(t, s)) < 0 \Longleftrightarrow \dot{c}(t), v(t), \dot{v}(t) \text{ are linearly independent.}$$

3.5 Show that the hyperbolic paraboloid of Example 3.3.3 has $K < 0$.

3.6 Show that every generalized cylinder and every generalized cone has $K = 0$.

3.7 Let S be the torus of Example 3.4.1. Find all points $p \in S$ where $K(p) > 0$ and all points where $K(p) < 0$.

3.8 Classify surfaces of revolution with constant Gauss curvature $K > 0$ or $K = 0$, i.e. state and prove the analogues of Propositions 3.4.2 and 3.4.3 in these cases.

3.9 (For readers familiar with covering spaces) Let $S \subset \mathbb{R}^3$ be non-empty, compact, connected, orientable regular surface with $K > 0$. Show that S is diffeomorphic to S^2.

Hint: Choose a smooth unit normal field $N : S \to S^2$. Verify that N is a covering map. Recall that S^2 is simply connected.

Chapter 4

The covariant derivative

To any pair X, Y of smooth vector fields on a manifold $M \subset \mathbb{R}^n$ we will introduce two new smooth vector fields on M, the *Lie bracket* $[X, Y]$ and the *covariant derivative* $\nabla_X Y$. The latter provides a way to differentiate Y with respect to X, and its definition involves the scalar product on $\mathbb{R}^n$. The covariant derivative will be used to define the Riemannian curvature tensor in Chapter 5. A main result of the present chapter says that the covariant derivative is *intrinsic* in the sense that it can be expressed in terms of the first fundamental form. This fact will be essential for the proof of Theorema Egregium.

The Lie bracket $[X, Y]$ agrees with $\nabla_X Y - \nabla_Y X$, but its definition does not refer to the scalar product on $\mathbb{R}^n$.

4.1 Directional derivatives

For any vector field X on a manifold M and smooth function $h : M \to \mathbb{R}^n$, the **directional derivative**

$$\partial_X h : M \to \mathbb{R}^n$$

is defined by

$$(\partial_X h)(p) := (dh_p)(X_p).$$

Proposition 4.1.1 *If X is a smooth vector field on the manifold M and $h : M \to \mathbb{R}^n$ is smooth then the directional derivative $\partial_X h$ is also smooth.*

Proof. Suppose $M \subset \mathbb{R}^m$. Given $p \in M$, we can find a neighbourhood W of p in $\mathbb{R}^m$ and smooth functions

$$\tilde{X} : W \to \mathbb{R}^m, \quad \tilde{h} : W \to \mathbb{R}^n$$

such that on $M \cap W$ we have $\tilde{X} = X$ and $\tilde{h} = h$. Let $\tilde{X}$ have components $(\tilde{X}^1, \ldots, \tilde{X}^m)$. For any point $q \in M \cap W$ we have

$$\partial_X h(q) = dh_q(X_q) = d\tilde{h}_q(\tilde{X}_q) = \sum_i \tilde{X}^i(q) \cdot \partial_i \tilde{h}(q).$$

Since the functions $\tilde{X}^i$ and $\partial_i \tilde{h}$ are smooth, we conclude that $\partial_X h$ is smooth on $M \cap W$. $\square$

Lemma 4.1.1 *Let X and Y be vector fields on a manifold M. For any smooth functions $f : M \to \mathbb{R}$, and $g, h : M \to \mathbb{R}^n$ the following hold.*

(i) $\partial_X(g + h) = \partial_X g + \partial_X h$

(ii) $\partial_X(fh) = (\partial_X f)h + f\partial_X h$

(iii) $\partial_{X+Y} h = \partial_X h + \partial_Y h$

(iv) $\partial_{fX} h = f\partial_X h$

Proof. Parts (i) and (ii) are left as exercises for the reader. Parts (iii) and (iv) follow from the linearity of the differential dh_p at any point $p \in M$. For instance,

$$(\partial_{fX} h)(p) = dh_p(f(p)X(p)) = f(p) \cdot dh_p(X(p)) = (f\partial_X h)(p). \qquad \square$$

Lemma 4.1.2 *Let M be a k–manifold. For any smooth map $h : M \to \mathbb{R}^n$ and local parametrization $F : U \to M$ with coordinate vector fields $X_1, \ldots, X_k$ the following holds for any i, j.*

(i) $(\partial_{X_i} h) \circ F = \partial_i(h \circ F)$.

(ii) $(\partial_{X_i} \partial_{X_j} h) \circ F = \partial_i \partial_j (h \circ F)$.

(iii) $(\partial_{X_i} X_j) \circ F = \partial_i \partial_j F$.

Proof. (i) For $u \in U$ and $p = F(u)$ we have

$$(\partial_{X_i} h)(p) = dh_p(X_i(p)) = dh_p(\partial_i F(u)) = \partial_i(h \circ F)(u),$$

where the last equality follows from the chain rule.

(ii) Applying (i) twice we get

$$(\partial_{X_i} \partial_{X_j} h) \circ F = \partial_i((\partial_{X_j} h) \circ F) = \partial_i \partial_j (h \circ F).$$

(iii) Take $h = X_j$ in (i). $\square$

Corollary 4.1.1 $\partial_{X_i} X_j = \partial_{X_j} X_i$.

Proof. This follows from part (iii) of the lemma because $\partial_i \partial_j F = \partial_j \partial_i F$. $\square$

4.2 The Lie bracket

Given smooth vector fields X, Y on a manifold M, the directional derivative $\partial_X Y$ will in general not be a vector field on M. However, the **Lie bracket**

$$[X, Y] := \partial_X Y - \partial_Y X \tag{4.1}$$

turns out to be a vector field. This is a consequence of the following proposition, which tells us how to compute the Lie bracket in local coordinates.

Proposition 4.2.1 *Let X, Y be smooth vector fields on a k–manifold M. If $X_1, \ldots, X_k$ are coordinate vector fields on an open subset V of M and*

$$X|_V = \sum_i a^i X_i, \quad Y|_V = \sum_j b^j X_j, \tag{4.2}$$

for (smooth) real-valued functions a^i, b^j on V then

$$[X, Y]|_V = \sum_{ij} (a^i \partial_{X_i} b^j - b^i \partial_{X_i} a^j) X_j.$$

Proof. We calculate

$$(\partial_X Y)|_V = \sum_{ij} a^i \partial_{X_i}(b^j X_j) = \sum_{ij} ((a^i \partial_{X_i} b^j) X_j + a^i b^j \partial_{X_i} X_j).$$

Applying Corollary 4.1.1 to $\partial_X Y - \partial_Y X$, the terms involving directional derivatives of the coordinate vector fields cancel out, and we obtain the formula in the lemma. □

Example 4.2.1 By Corollary 4.1.1, one has

$$[X_i, X_j] = 0$$

whenever $X_1, \ldots, X_k$ are coordinate vector fields on an open set in M.

Proposition 4.2.2 *For any smooth vector fields X, Y on a manifold M and smooth function $f : M \to \mathbb{R}$ one has*

$$\partial_{[X,Y]} f = \partial_X \partial_Y f - \partial_Y \partial_X f. \tag{4.3}$$

Proof. In a neighbourhood of any point in M we can express X and Y in terms of coordinate vector fields as in (4.2). In that neighbourhood we then have

$$\partial_X \partial_Y f = \sum_{ij} a^i \partial_{X_i}(b^j \partial_{X_j} f) = \sum_{ij} (a^i \partial_{X_i} b^j \cdot \partial_{X_j} f + a^i b^j \partial_{X_i} \partial_{X_j} f).$$

By Lemma 4.1.2 (ii) we have $\partial_{X_i}\partial_{X_j} f = \partial_{X_j}\partial_{X_i} f$. Applying this to $\partial_X \partial_Y f - \partial_Y \partial_X f$, the terms involving second order directional derivatives cancel out. Comparing the resulting formula with the expression in Proposition 4.2.1 we obtain (4.3). □

Proposition 4.2.3 *Let X, Y be smooth vector fields on a manifold M and $f : M \to \mathbb{R}$ a smooth function.*

(i) $[fX, Y] = f[X, Y] - (\partial_Y f)X$.

(ii) $[X, fY] = f[X, Y] + (\partial_X f)Y$.

Proof. See Problem 4.2. □

4.3 The covariant derivative

Let $M \subset \mathbb{R}^n$ be a manifold. For any $p \in M$ let

$$\Pi_p : \mathbb{R}^n \to T_p M$$

be the orthogonal projection. Given a function $f : M \to \mathbb{R}^n$, the **tangential part** of f is the vector field $f^{\tan}$ on M defined by

$$f^{\tan}(p) := \Pi_p(f(p)).$$

Proposition 4.3.1 *If $f : M \to \mathbb{R}^m$ is smooth then the tangential part f^{tan} is also smooth.*

Proof. Given $p \in M$, choose a smooth frame on some neighbourhood V of p in M. Applying the Gram-Schmidt process to this frame produces a smooth **orthonormal frame** $(E_1, \ldots, E_d)$ on V, where "orthonormal" means that $(E_1(x), \ldots, E_d(x))$ is an orthonormal basis for $T_x M$ for every $x \in V$. On V one then has

$$f^{\tan} = \sum_{i=1}^{d} \langle f, E_i \rangle E_i,$$

proving that $f^{\tan}$ is smooth on V. $\square$

Example 4.3.1 Given $p \in M$ and $v \in T_pM$, let $h : M \to \mathbb{R}^d$ be the constant map $h \equiv v$. Then $X := h^{\tan}$ is a smooth vector field on M with $X_p = v$.

If X, Y are smooth vector fields on M then the **covariant derivative** $\nabla_X Y$ is the smooth vector field on M defined by

$$\nabla_X Y := (\partial_X Y)^{\tan}.$$

One can also define the covariant derivative at a point: If $p \in M$ and $v \in T_pM$ then we define

$$\nabla_{p,v} Y := \Pi_p(dY_p(v)).$$

If $v = X_p$ we therefore have $(\nabla_X Y)(p) = \nabla_{p,v} Y$.

Proposition 4.3.2 *For any smooth vector fields X, Y, Z on M and smooth function $f : M \to \mathbb{R}$ one has*

(i) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

(ii) $\nabla_X(fY) = (\partial_X f) \cdot Y + f\nabla_X Y$

(iii) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

(iv) $\nabla_{fX} Z = f\nabla_X Z$

(v) $\partial_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

Proof. Left to the reader. $\square$

Proposition 4.3.3 *For any smooth vector fields X and Y on M one has*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proof. Take tangential parts on both sides in Definition 4.1. $\square$

Proposition 4.3.4 *If $\{X_i\}$ are coordinate vector fields on an open set in M then*

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i.$$

Proof. This follows from Corollory 4.1.1 by taking tangential parts. □

Let $(X_1, \ldots, X_d)$ be a coordinate frame on an open subset $V \subset M$. Recall that $(X_1(p), \ldots, X_d(p))$ is a basis for the tangent space T_pM for every $p \in V$. Any vector field X on V can therefore be expressed uniquely on the form

$$X = \sum_{i=1}^{d} a^i X_i$$

for some functions $a^i : V \to \mathbb{R}$. In view of Proposition 4.3.2, the covariant derivative on V is therefore complete determined by the collection of vector fields $\nabla_{X_i} X_j$. On the other hand,

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k \tag{4.4}$$

for some smooth functions $\Gamma_{ij}^k : V \to \mathbb{R}$ called **Christoffel symbols**. Note that

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

by Proposition 4.3.4.

Recall that the components of the first fundamental form are

$$g_{ij} = \langle X_i, X_j \rangle.$$

Let (g^{ij}) be the inverse matrix of the 2×2 matrix (g_{ij}), so that

$$\sum_j g_{ij} g^{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{else.} \end{cases}$$

Proposition 4.3.5 $\Gamma_{ij}^k = \dfrac{1}{2} \displaystyle\sum_\ell g^{k\ell} (\partial_{X_i} g_{j\ell} + \partial_{X_j} g_{i\ell} - \partial_{X_\ell} g_{ij})$.

Proof. We calculate

$$\begin{aligned} \partial_{X_i} g_{jk} &= \partial_{X_i} \langle X_j, X_k \rangle \\ &= \langle \nabla_i X_j, X_k \rangle + \langle X_j, \nabla_i X_k \rangle \\ &= \langle \sum_m \Gamma_{ij}^m X_m, X_k \rangle + \langle X_j, \sum_m \Gamma_{ik}^m X_m \rangle \\ &= \sum_m \left(\Gamma_{ij}^m \, g_{mk} + \Gamma_{ik}^m \, g_{jm} \right). \end{aligned}$$

We now make cyclic permutations of the indices i, j, k to obtain three equations:

$$\begin{aligned}
\partial_{X_i} g_{jk} &= \sum_m \left(\Gamma_{ij}^m\, g_{mk} + \Gamma_{ik}^m\, g_{jm}\right), \\
\partial_{X_j} g_{ki} &= \sum_m \left(\Gamma_{jk}^m\, g_{mi} + \Gamma_{ji}^m\, g_{km}\right), \\
\partial_{X_k} g_{ij} &= \sum_m \left(\Gamma_{ki}^m\, g_{mj} + \Gamma_{kj}^m\, g_{im}\right).
\end{aligned}$$

Adding the first two equations and subtracting the last one we see that four terms cancel and we are left with

$$\partial_{X_i} g_{jk} + \partial_{X_j} g_{ki} - \partial_{X_k} g_{ij} = 2 \sum_m \Gamma_{ij}^m\, g_{mk},$$

which yields

$$\Gamma_{ij}^k = \sum_{\ell m} \Gamma_{ij}^m g^{k\ell} g_{\ell m} = \frac{1}{2} \sum_{\ell} g^{kl} (\partial_{X_i} g_{jl} + \partial_{X_j} g_{il} - \partial_{X_\ell} g_{ij}). \qquad \square$$

Proposition 4.3.6 *Let $S \subset \mathbb{R}^3$ be a regular surface with local parametrization $F : U \to S$ and corresponding coordinate vector fields X_1, X_2. Let N be a smooth unit normal field on $F(U)$. If $u \in U$ and $p = F(u)$ then*

$$\partial_i \partial_j F(u) = \sum_k \Gamma_{ij}^k(p) X_k(p) + h_{ij}(p) N(p),$$

where (h_{ij}) are the components of the second fundamental form.

Proof. Expressing $\partial_{X_i} X_j$ in terms of its tangential and normal parts we get

$$\begin{aligned}
\partial_i \partial_j F(u) &= (\partial_{X_i} X_j)(p) \\
&= (\nabla_{X_i} X_j)(p) + \langle \partial_i \partial_j F(u), N(p) \rangle N(p) \\
&= \sum_k \Gamma_{ij}^k(p) X_k(p) + h_{ij}(p) N(p). \qquad \square
\end{aligned}$$

Problems

4.1 Let M be a manifold and $f : M \to \mathbb{R}$ a smooth function such that $\partial_X f = 0$ for all vector fields X. Prove that f is locally constant, and f is constant if M is connected.

4.2 Prove Proposition 4.2.3.

4.3 Let $E \subset \mathbb{R}^n$ be an affine subspace (cf. Problem 1.5). Let X, Y be smooth vector fields on E. Show that $\nabla_X Y = \partial_X Y$.

4.4 Let $\phi : M \to N$ be a local isometry between manifolds, and let X, Y be smooth vector fields on N. Show that

$$\phi^*(\nabla_X Y) = \nabla_{\phi^* X}(\phi^* Y),$$

where pull-back of vector fields by ϕ is as described in Problem 1.17.

Chapter 5

The Riemannian curvature tensor

In this chapter we introduce the Riemannian curvature tensor, which is defined for any manifold, and prove that it is preserved by local isometries. For regular surfaces in $\mathbb{R}^3$ we will express the Gauss curvature in terms of the Riemannian curvature tensor and deduce that the Gauss curvature is also preserved by local isometries. The latter result is the *Theorema Egregium* (remarkable theorem) of Gauss.

5.1 Some algebra

Let $E_1, \ldots, E_k, F$ be modules over a ring R. A map

$$T : E_1 \times \cdots \times E_k \to F$$

is called R–**multilinear** (or **multilinear over** R) if it is linear in each variable separately, i.e. if for any $a_i \in E_i$, $i = 1, \ldots, k$ and index j the map

$$E_j \to F, \quad b \mapsto T(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_k)$$

is R–linear.

For any manifold M, the collection $C^\infty(M)$ of all smooth functions $M \to \mathbb{R}$ is a commutative ring where addition and multiplication are defined pointwise: If $f, g \in C^\infty(M)$ and $p \in M$ then

$$(f + g)(p) = f(p) + g(p), \quad (fg)(p) = f(p)g(p).$$

An example of a module over $C^\infty(M)$ is the collection $\mathfrak{X}(M)$ of all smooth vector fields on M, where addition of vector fields as well as multiplication of a vector field with a function are defined pointwise.

5.2 The Riemannian curvature tensor

As motivation, we first consider the case when M is an affine subspace of $\mathbb{R}^n$. Then $\nabla_X Y = \partial_X Y$ for any smooth vector fields X, Y on M. If Z is a third smooth vector field on M then by applying Proposition 4.2.2 to each component of Z we get

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \partial_X \partial_Y Z - \partial_Y \partial_X Z = \partial_{[X,Y]} Z = \nabla_{[X,Y]} Z.$$

For an arbitrary manifold M, the **Riemannian curvature tensor** associates to every triple X, Y, Z of smooth vector fields on M the smooth vector field

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Thus, if M is an affine space then $R = 0$. We are going to show that the Riemannian curvature tensor is preserved by local isometries, hence it provides a measure of how much a given manifold deviates from being locally isometric to a Euclidean space. We will also express the Gauss curvature K in terms of R, proving that Gauss curvature is also preserved by local isometries.

Proposition 5.2.1 *The map*

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X,Y,Z) \mapsto R(X,Y)Z$$

is multilinear over $C^\infty(M)$.

Proof. This is a straightforward application of Propositions 4.2.3 and 4.3.2. Additivity in each variable is obvious. Now let $f \in C^\infty(M)$. Then

$$\begin{aligned} R(fX,Y)Z &= f\nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{f[X,Y] - \partial_Y f \cdot X}(Z) \\ &= f\nabla_X \nabla_Y Z - \partial_Y f \cdot \nabla_X Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z + \partial_Y f \cdot \nabla_X Z \\ &= f R(X,Y)Z. \end{aligned}$$

Furthermore,

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y,X)Z = fR(X,Y)Z.$$

The proof that $R(X,Y)(fZ) = fR(X,Y)Z$ is left as an exercise. □

Proposition 5.2.2 *For all smooth vector fields X, Y, Z, W on M one has*

$$\langle R(X,Y)Z, W \rangle = -\langle Z, R(X,Y)W \rangle.$$

Proof. We calculate

$$\begin{aligned}\partial_X\partial_Y\langle Z,W\rangle &= \partial_X(\langle\nabla_Y,W\rangle + \langle Z,\nabla_Y,W\rangle)\\ &= \langle\nabla_X\nabla_Y Z,W\rangle + \langle\nabla_Y Z,\nabla_X W\rangle\\ &+ \langle\nabla_X Z,\nabla_Y W\rangle + \langle Z,\nabla_X\nabla_Y W\rangle).\end{aligned}$$

In the final expression, the sum of the second and third terms is symmetric in X and Y. If we make the same calculation with X and Y reversed and subtract the results, the terms involving first-order covariant derivatives therefore cancel out, and we obtain the following.

$$\begin{aligned}\partial_{[X,Y]}\langle Z,W\rangle &= \partial_X\partial_Y\langle Z,W\rangle - \partial_Y\partial_X\langle Z,W\rangle\\ &= \langle\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z,W\rangle + \langle Z,\nabla_X\nabla_Y W - \nabla_Y\nabla_X W\rangle.\end{aligned}$$

Combining this with

$$\partial_{[X,Y]}\langle Z,W\rangle = \langle\nabla_{[X,Y]}Z,W\rangle + \langle Z,\nabla_{[X,Y]}W\rangle$$

we obtain the proposition. □

The previous proposition makes it possible to define the Riemannian curvature R_p at any point p in M, as we now explain. For any finite-dimensional real vector space V equipped with a scalar product let $\mathrm{so}(V)$ denote the space of all skew-symmetric endomorphisms of V, i.e. linear maps $A : V \to V$ such that

$$\langle Ax,y\rangle = -\langle x,Ay\rangle$$

for all $x,y \in V$.

Proposition 5.2.3 *Let M be a manifold. For each point $p \in M$ there is a unique skew-symmetric bilinear map*

$$R_p : T_pM \times T_pM \to so(T_p(M))$$

such that for any smooth vector fields X,Y,Z on M one has

$$[R(X,Y)Z]_p = R_p(X_p,Y_p)Z_p.$$

Proof. The map R_p is unique because for every tangent vector $v \in T_pM$ there exists a smooth vector field X on M such that $X_p = v$ (see Example 4.3.1). To prove existence, let $(X_1,\ldots,X_d)$ be a smooth frame in some neighbourhood V of p in M. Let

$$\tilde{R}_p : T_pM \times T_pM \times T_pM \to T_pM$$

be the unique multilinear map such that for $i, j, k = 1, \ldots, d$ one has

$$\tilde{R}_p(X_i(p), X_j(p), X_k(p)) = [R(X_i, X_j))X_k](p).$$

For any $u, v, w \in T_pM$ we define

$$R_p(u, v)w := \tilde{R}_p(u, v, w).$$

Now let X, Y, Z be smooth vector fields on M. Over V we can express these as

$$X|_V = \sum_i a^i X_i, \quad Y|_V = \sum_j b^j X_j, \quad Z|_V = \sum_k c^k X_k,$$

where a^i, b^j, c^k are smooth functions on V. Then

$$R(X, Y)Z = \sum_{ijk} a^i b^j c^k R(X_i, X_j)X_k$$

by Proposition 5.2.1, so

$$\begin{aligned}[R(X, Y)Z]_p &= \sum_{ijk} a^i(p)b^j(p)c^k(p)R_p(X_i(p), X_j(p))X_k(p) \\ &= R_p(X_p, Y_p)Z_p.\end{aligned}$$

It now follows from Proposition 5.2.2 that $R_p(u, v)$ is skew-symmetric for all $u, v \in T_pM$. $\square$

Given a coordinate frame $(X_1, \ldots, X_d)$ on an open subset V of a manifold M, there are smooth, real-valued functions R^ℓ_{ijk} on V such that

$$R(X_i, X_j)X_k = \sum_\ell R^\ell_{ijk} X_\ell.$$

These functions R^ℓ_{ijk} are called the *components* of the curvature tensor.

Proposition 5.2.4 $R^\ell_{ijk} = \partial_{X_i}\Gamma^\ell_{jk} - \partial_{X_j}\Gamma^\ell_{ik} + \sum_m \left(\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}\right)$.

Proof. This is a straight-forward calculation:

$$\begin{aligned}\nabla_{X_i}\nabla_{X_j}X_k &= \nabla_{X_i}\sum_\ell \Gamma^\ell_{jk}X_\ell \\ &= \sum_\ell \left(\partial_{X_i}\Gamma^\ell_{jk}\cdot X_\ell + \Gamma^\ell_{jk}\nabla_{X_i}X_\ell\right) \\ &= \sum_\ell \partial_{X_i}\Gamma^\ell_{jk}\cdot X_\ell + \sum_{\ell m}\Gamma^\ell_{jk}\Gamma^m_{i\ell}X_m.\end{aligned}$$

Interchanging ℓ and m in the last sum we obtain

$$\nabla_{X_i}\nabla_{X_j}X_k = \sum_\ell \left(\partial_{X_i}\Gamma^\ell_{jk} + \sum_m \Gamma^m_{jk}\Gamma^\ell_{im}\right) X_\ell.$$

Now recall that $[X_i, X_j] = 0$, hence

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k.$$

Putting this together, we get the formula of the proposition. □

We now prove that the Riemannian curvature tensor is preserved by local isometries.

Theorem 5.2.1 *Let $\phi : M \to \bar{M}$ be a local isometry between manifolds with curvature tensors R and $\bar{R}$, respectively. Let $p \in M$, $\bar{p} := \phi(p)$, and $A := d\phi_p$. Then for all tangent vectors $v_1, v_2, v_3 \in T_pM$ one has*

$$A(R_p(v_1, v_2)v_3) = \bar{R}_{\bar{p}}(Av_1, Av_2)Av_3.$$

Proof. Let $F : U \to M$ be a local parametrization of M around p. After shrinking U, if necessary, the map $\bar{F} := \phi \circ F$ will be a local parametrization of $\bar{M}$. Let X_i, g_{ij}, Γ^k_{ij}, and R^ℓ_{ijk} be the coordinate vector fields, components of the first fundamental form, Christoffel symbols, and components of the curvature tensor defined by F. Let $\bar{X}_i$, $\bar{g}_{ij}$, $\bar{\Gamma}^k_{ij}$, and $\bar{R}^\ell_{ijk}$, be the corresponding quantities defined by $\bar{F}$.

Suppose $p = F(u)$, $u \in U$. The chain rule yields

$$AX_i(p) = d\phi_p(\partial_i F(u)) = \partial_i(\phi \circ F)(u) = \partial_i \bar{F}(u) = \bar{X}_i(\bar{p}).$$

Because the differential $A = d\phi_p$ is an isometry,

$$g_{ij}(p) = \langle X_i(p), X_j(p)\rangle = \langle AX_i(p), AX_j(p)\rangle = \langle \bar{X}_i(\bar{p}), \bar{X}_j(\bar{p})\rangle = \bar{g}_{ij}(\bar{p}).$$

Proposition 4.3.5 then implies that $\Gamma^k_{ij}(p) = \bar{\Gamma}^k_{ij}(\bar{p})$, and Proposition 5.2.4 yields $R^\ell_{ijk}(p) = \bar{R}^\ell_{ijk}(\bar{p})$. Given tangent vectors $v_1, v_2, v_3 \in T_pM$, the equation

$$A(R_p(v_1, v_2)v_3) = \bar{R}_{\bar{p}}(Av_1, Av_2)Av_3$$

therefore holds whenever each v_i is one of the basis vectors $X_j(p)$. By multilinearity of R_p, the same equation holds for all v_i. □

5.3 Theorema Egregium

Our next goal is to express the Gauss curvature of a regular surface in terms of the Riemannian curvature tensor. This will lead to a proof of Gauss's *Theorema Egregium*, which asserts that the Gauss curvature is preserved by local isometries.

Let $M \subset \mathbb{R}^n$ be a manifold. The **normal part** of a function $f : M \to \mathbb{R}^n$ is the normal field f^{nor} on M defined by

$$f^{\mathrm{nor}} := f - f^{\mathrm{tan}}.$$

Given smooth vector fields X, Y on M we define the normal field

$$\alpha(X, Y) := (\partial_X Y)^{\mathrm{nor}},$$

so that

$$\partial_X Y = \nabla_X Y + \alpha(X, Y)$$

is the decomposition of $\partial_X Y$ into its tangential and normal parts.

Proposition 5.3.1 $\alpha(X, Y) = \alpha(Y, X)$.

Proof. By definition of the Lie bracket we have

$$[X, Y] = \partial_X Y - \partial_Y X.$$

Because $[X, Y]$ is a vector field, its normal part is zero, hence

$$0 = [X, Y]^{\mathrm{nor}} = (\partial_X Y)^{\mathrm{nor}} - (\partial_Y X)^{\mathrm{nor}} = \alpha(X, Y) - \alpha(Y, X). \qquad \square$$

Let $\mathfrak{X}^{\perp}(M)$ be the set of all smooth normal fields on M, which is a module over the ring $C^\infty(M)$ of smooth functions on M.

Proposition 5.3.2 *The map*

$$\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^{\perp}(M)$$

is bilinear over $C^\infty(M)$.

Proof. Biadditivity of α is obvious. For $f \in C^\infty(M)$ we have

$$\alpha(fX, Y) = (\partial_{fX} Y)^{\mathrm{nor}} = (f\partial_X Y)^{\mathrm{nor}} = f\alpha(X, Y).$$

By symmetry of α we also have $\alpha(X, fY) = f\alpha(X, Y)$. $\square$

Proposition 5.3.3 *Let M be a manifold. For any point $p \in M$ there is a unique symmetric bilinear map*

$$\alpha_p : T_pM \times T_pM \to (T_pM)^\perp$$

such that if X, Y are smooth vector fields defined in some neighbourhood of p in M then

$$[\alpha(X,Y)]_p = \alpha_p(X_p, Y_p).$$

Proof. This is proved in the same way as the corresponding statement for the Riemannian curvature tensor, see Proposition 5.2.3. □

The bilinear form α_p on T_pM is called the **second fundamental form**. The following proposition explains how this notion generalizes the second fundamental form as defined earlier for regular surfaces in $\mathbb{R}^3$ equipped with a smooth unit normal field.

Proposition 5.3.4 *Let $S \subset \mathbb{R}^3$ be a regular surface. If $N : S \to \mathbb{R}^3$ is a smooth unit normal field and $p \in S$ then for all tangent vectors $v, w \in T_pS$ one has*

$$\alpha_p(v,w) = I\!I_p(v,w) \cdot N_p,$$

where $I\!I_p$ is the second fundamental form relative to N.

Proof. Choose smooth vector fields X, Y defined in some neighbourhood of p in S such that $X_p = v$ and $Y_p = w$. Then

$$0 = \partial_X \langle Y, N\rangle = \langle \partial_X Y, N\rangle + \langle Y, \partial_X N\rangle.$$

Evaluating at p we get

$$\begin{aligned}\langle \alpha_p(v,w), N_p\rangle &= \langle \partial_X Y, N\rangle_p = -\langle Y, \partial_X N\rangle_p \\ &= -\langle w, dN_p(v)\rangle = \langle w, W_p(v)\rangle = I\!I_p(v,w),\end{aligned}$$

where W_p is the Weingarten map. □

Theorem 5.3.1 (Gauss equation) *If X, Y, Z, W are smooth vector fields on a manifold M then*

$$-\langle R(X,Y)Z, W\rangle = \langle \alpha(X,Z), \alpha(Y,W)\rangle - \langle \alpha(X,W), \alpha(Y,Z)\rangle.$$

Proof. We begin by calculating

$$\begin{aligned}\langle \partial_X \partial_Y Z, W\rangle &= \langle \partial_X(\nabla_Y Z + \alpha(Y,Z)), W\rangle \\ &= \langle \nabla_X \nabla_Y, W\rangle + \langle \partial_X \alpha(Y,Z), W\rangle.\end{aligned}$$

On the other hand,

$$0 = \partial_X \langle \alpha(Y,Z), W\rangle = \langle \partial_X \alpha(Y,Z), W\rangle + \langle \alpha(Y,Z), \partial_X W\rangle,$$

hence

$$\langle \partial_X \alpha(Y,Z), W\rangle = -\langle \alpha(Y,Z), \alpha(X,W)\rangle.$$

Altogether, we obtain

$$\langle \partial_X \partial_Y Z, W\rangle = \langle \nabla_X \nabla_Y, W\rangle - \langle \alpha(Y,Z), \alpha(X,W)\rangle.$$

Finally,

$$\begin{aligned}\langle \nabla_{[X,Y]} Z, W\rangle &= \langle \partial_{[X,Y]} Z, W\rangle \\ &= \langle \partial_X \partial_Y Z - \partial_Y \partial_X Z, W\rangle \\ &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W\rangle \\ &- \langle \alpha(Y,Z), \alpha(X,W)\rangle + \langle \alpha(X,Z), \alpha(Y,W)\rangle,\end{aligned}$$

from which the theorem follows. □

Theorem 5.3.2 *Let $S \subset \mathbb{R}^3$ be a regular surface with Gauss curvature K and Riemannian curvature tensor R. Then for any $p \in S$ and orthonormal basis v_1, v_2 for T_pS one has*

$$K(p) = -\langle R_p(v_1, v_2)v_1, v_2\rangle. \tag{5.1}$$

Proof. Choose a smooth unit normal field N on a neighbourhood of p in S. Let (a_{ij}) be the matrix of the Weingarten map $W_p : T_pS \to T_pS$ with respect to the basis v_1, v_2. Then

$$a_{ij} = \langle v_i, W_p(v_j)\rangle = I\!I(v_i, v_j).$$

By Proposition 5.3.4 we have

$$\alpha_p(v_i, v_j) = a_{ij} N_p.$$

Since $\langle N_p, N_p\rangle = 1$, the Gauss equation yields

$$-\langle R_p(v_1, v_2)v_1, v_2\rangle = a_{11}a_{22} - a_{12}^2 = \det W_p = K(p). \qquad \square$$

Theorem 5.3.3 (Theorema Egregium) *If $\phi : S \to \bar{S}$ is a local isometry between regular surfaces in $\mathbb{R}^3$ with Gauss curvatures $K, \bar{K}$, respectively, then*

$$K = \bar{K} \circ \phi.$$

Proof. Let $p \in S$ and $\bar{p} = \phi(p)$. We must show that $K(p) = \bar{K}(\bar{p})$.

Let $A := d\phi_p$. If v_1, v_2 is an orthonormal basis for T_pS, then Av_1, Av_2 is an orthonormal basis for $T_{\bar{p}}\bar{S}$, and by Theorems 5.2.1 and 5.3.2 we have

$$\begin{aligned} K(p) &= -\langle R_p(v_1, v_2)v_1, v_2\rangle = -\langle A(R_p(v_1, v_2)v_1), Av_2\rangle \\ &= -\langle R_p(Av_1, Av_2)Av_1, Av_2\rangle = \bar{K}(\bar{p}). \qquad \square \end{aligned}$$

Theorem 5.3.2 can be used to generalize Gauss curvature to regular surfaces $S \subset \mathbb{R}^n$ for any $n \geq 2$. Given $p \in S$, we define $K(p)$ by the formula (5.1), where v_1, v_2 is any orthonormal basis for T_pS. It is easy to see that $K(p)$ is independent of the choice of v_1, v_2 (see Problem 5.1). With this definition, Theorem 5.3.3 extends to local isometries between regular surfaces in Euclidean spaces of any dimension.

Problems

5.1 Let M be a manifold, $p \in M$, and let $E \subset T_pM$ be a 2–dimensional linear subspace. The **sectional curvature** $\sec_p(E)$ is defined by

$$\sec_p(E) := \langle R_p(v_1, v_2)v_1, v_2\rangle,$$

where v_1, v_2 is an orthonormal basis for T_pS. Show that $\sec_p(E)$ is independent of the choice of v_1, v_2.

5.2 A smooth vector field Y on a manifold M is called **parallel** or **covariantly constant** if $\nabla_X Y = 0$ for all vector fields X on M. Prove the following.

(a) If Y is parallel then the function $p \mapsto \|Y_p\|$ on M is locally constant.

(b) If a regular surface S admits a non-zero parallel vector field Y then the Gauss curvature of S vanishes.

Hint: By rescaling Y we obtain a parallel vector field E_1 of unit length. Locally, one can extend E_1 to a smooth orthonormal frame (E_1, E_2). Now apply Theorem 5.3.2.

(c) If (E_1, E_2) is an orthonormal frame on a regular surface S and E_1 is parallel, then E_2 is parallel, too.

Hint: Consider $\partial_X\langle E_i, E_j\rangle$ for arbitrary vector fields X.

Chapter 6

Geodesics

6.1 Parallel transport

Let $M \subset \mathbb{R}^n$ be a manifold and $\gamma : I \to M$ a parametrized curve. A **vector field on** M **along** γ is a map $W : I \to \mathbb{R}^n$ such that $W(t) \in T_{\gamma(t)}M$ for all $t \in I$.

Example 6.1.1 If X is a vector field on M then $X \circ \gamma$ is a vector field on M along γ.

Example 6.1.2 If $\gamma : I \to M$ a smooth parametrized curve then $\dot{\gamma}$ is a vector field on M along γ.

If W is a smooth vector field on M along γ then the **covariant derivative** of W is the vector field $\frac{\nabla}{dt}W = \frac{\nabla W}{dt}$ along γ defined by

$$\frac{\nabla}{dt}W(t) := \Pi_{\gamma(t)}\dot{W}(t).$$

Here, $\Pi_p : \mathbb{R}^n \to T_pM$ is the orthogonal projection, for any $p \in M$. As in the proof of Proposition 4.3.1 we find that $\frac{\nabla}{dt}W$ is smooth if γ is smooth. If $\frac{\nabla}{dt}W = 0$ then W is called **parallel** or **covariantly constant**.

Proposition 6.1.1 *Let W, Z be smooth vector fields along $\gamma : I \to M$, and let $f : I \to \mathbb{R}$ be smooth. Then the following hold.*

(i) $\frac{\nabla}{dt}(W + Z) = \frac{\nabla}{dt}W + \frac{\nabla}{dt}Z$.

(ii) $\frac{\nabla}{dt}(fW) = \dot{f}W + f\frac{\nabla}{dt}W$.

(iii) $\frac{d}{dt}\langle W(t), Z(t)\rangle = \langle \frac{\nabla}{dt}W(t), Z(t)\rangle + \langle W(t), \frac{\nabla}{dt}Z(t)\rangle$.

Proof. Left to the reader. □

The following proposition relates covariant derivatives of vector fields to covariant derivatives of vector fields along curves.

Proposition 6.1.2 *Let Y be a smooth vector field on a manifold M and $\gamma : I \to M$ a smooth curve. Let $t_0 \in I$, $p := \gamma(t_0)$, and $v := \dot{\gamma}(t_0)$. Then*

$$\nabla_{p,v} Y = \frac{\nabla(Y \circ \gamma)}{dt}(t_0).$$

Proof. We have

$$\begin{aligned}\frac{\nabla(Y \circ \gamma)}{dt}(t_0) &= \Pi_p \left.\frac{d}{dt}\right|_{t_0} Y(\gamma(t)) \\ &= \Pi_p dY_p(v) \\ &= \nabla_{p,v} Y \qquad \square\end{aligned}$$

We will now express the covariant derivative in terms of a local parametrization. Let W be a smooth vector field on a d–manifold M along a smooth curve $\gamma : I \to M$. Suppose $F : U \to M$ is a local parametrization whose image contains $\gamma(I)$. Let $(X_1, \ldots, X_d)$ be the corresponding coordinate frame. For any t we can express the tangent vector $W(t)$ in terms of the basis $\{X_i(\gamma(t))\}$ for $T_{\gamma(t)}M$. Thus, there are smooth real-valued functions $f^1, \ldots, f^d$ on I such that

$$W(t) = \sum_i f^i(t) X_i(\gamma(t)).$$

To write down the analogous formula for $\dot{\gamma}(t)$, we consider the composite curve $\alpha := F^{-1} \circ \gamma$ with values in U. Applying the chain rule to $\gamma = F \circ \alpha$ we get

$$\dot{\gamma}(t) = \sum_i \dot{\alpha}^i(t) \partial_i F(\alpha(t)) = \sum_i \dot{\alpha}^i(t) X_i(\gamma(t)),$$

where $\alpha^1, \ldots, \alpha^d$ are the components of α. By Proposition 6.1.1,

$$\frac{\nabla}{dt} W(t) = \sum_j \left(\dot{f}^j(t) X_j(\gamma(t)) + f^j(t) \frac{\nabla}{dt} X_j(\gamma(t)) \right).$$

Using Proposition 6.1.2 and recalling the definition (4.4) of the Christoffel

symbols, we get

$$\begin{aligned}\frac{\nabla}{dt}X_j(\gamma(t)) &= \nabla_{\gamma(t),\dot\gamma(t)}X_j \\ &= \sum_i \dot\alpha^i(t)\nabla_{\gamma(t),X_i(\gamma(t))}X_j \\ &= \sum_i \dot\alpha^i(t)\left(\nabla_{X_i}X_j\right)(\gamma(t)) \\ &= \sum_{jk} \dot\alpha^i(t)\Gamma^k_{ij}(\gamma(t))X_k(\gamma(t)).\end{aligned}$$

Putting this together we find

$$\frac{\nabla}{dt}W(t) = \sum_k \left(\dot f^k(t) + \sum_{ij}\Gamma^k_{ij}(\gamma(t))\dot\alpha^i(t)f^j(t)\right)X_k(\gamma(t)).$$

In particular, $\frac{\nabla}{dt}W = 0$ if and only if for all $t \in I$ one has

$$\dot f^k(t) + \sum_{ij}\tilde\Gamma^k_{ij}(\alpha(t))\dot\alpha^i(t)f^j(t) = 0, \quad k = 1,\dots,d, \tag{6.1}$$

where $\tilde\Gamma = \Gamma \circ F$. This is a system of first order, linear, ordinary differential equations. Therefore, for any $t_0 \in I$ and initial value $(a^1,\dots,a^d) \in \mathbb{R}^d$ there is a unique solution $\{f^i\}$ such that $f^i(t_0) = a^i$ for all i.

In the above discussion we assumed that the trace of the curve γ was contained in a single coordinate domain. In the following theorem we consider vector fields on M along an arbitrary smooth curve.

Theorem 6.1.1 *Let $\gamma : I \to M$ be a smooth curve and $t_0 \in I$, $v_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field W on M along γ such that $W(t_0) = v_0$.*

Proof. Case 1: The trace of γ is contained in a single coordinate domain. This was proved above.

Case 2: Arbitrary γ. We first prove the uniqueness part. Suppose W_1, W_2 are parallel vector fields along γ such that $W_i(t_0) = v_0$ for $i = 1, 2$. Let

$$A := \{t \in I \mid W_1(t) = W_2(t)\}.$$

By assumption, $t_0 \in A$. Because $\mathbb{R}^d$ is Hausdorff and each W_i is continuous, the set A is closed in I. The uniqueness part of Case 1 implies that A is also open in I. But I is connected, so $A = I$.

To prove existence, let B be the union of all open subintervals $J \subset I$ containing t_0 such that there exists a parallel vector field W along $\gamma|_J$ satisfying $W(t_0) = v_0$. Evidently, B is an open subinterval of I, and $t_0 \in B$ by Case 1. The uniqueness part which we already proved, ensures that there is a parallel vector field W_0 along all of γ such that $W_0(t_0) = v_0$. We must prove that $B = I$. Suppose on the contrary that $B \neq I$. Then B must have an end-point b which lies in I. Choose a coordinate domain V on M such that $\gamma(b) \in V$. Because γ is continuous there exists an open subinterval $I_1 \subset I$ containing b such that $\gamma(I_1) \subset V$. Since b is an end-point of B there exists a point $t_1 \in B \cap I_1$. By Case 1 there exists a parallel vector field W_1 along $\gamma|_{I_1}$ such that $W_1(t_1) = W_0(t_1)$. The uniqueness part of Case 1 now implies that $W_0 = W_1$ on $B \cap I_1$. Hence, W_0 and W_1 patch together to a parallel vector field W along $\gamma_{B \cup I_1}$ such that $W(t_0) = v_0$. But then b cannot be an end-point of B, a contradiction. This proves the existence part of the theorem. $\square$

Given a parametrized curve $\gamma : I \to M$ and $t_0, t_1 \in I$, the **parallel transport** or **holonomy** along γ from t_0 to t_1 is the map

$$\mathrm{Hol}^{\gamma}_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$

defined as follows. Given $v_0 \in T_{\gamma(t_0)}M$, let W be the unique parallel vector field on M along γ such that $W(t_0) = v_0$. Then

$$\mathrm{Hol}^{\gamma}_{t_0,t_1}(v_0) := W(t_1).$$

Proposition 6.1.3 *For any smooth curve $\gamma : I \to M$ on a manifold M and $t_0, t_1 \in I$ the following hold.*

(i) *$Hol^{\gamma}_{t_0,t_1}$ is linear isometry.*

(ii) *If W is a parallel vector field along γ and $u : J \to I$ is smooth, where J is an open interval, then $W \circ u$ is parallel along $\gamma \circ u$.*

(iii) *For any smooth vector field W along γ the covariant derivative of W can be expressed in terms of parallel transport by*

$$\left.\frac{\nabla}{dt}\right|_{t_0} W(t) = \left.\frac{d}{dt}\right|_{t_0} Hol^{\gamma}_{t,t_0}(W(t)).$$

Proof. (i) Let V, W be parallel vector fields along γ, and $a, b \in \mathbb{R}$. Then

$$\frac{\nabla}{dt}(aV + bW) = a\frac{\nabla}{dt}V + b\frac{\nabla}{dt}W = 0,$$

proving that the parallel transport $\mathrm{Hol}^{\gamma}_{t_0,t_1}$ is a linear map. Moreover,

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle \frac{\nabla}{dt}V(t), W(t)\rangle + \langle V(t), \frac{\nabla}{dt}W(t)\rangle = 0,$$

hence $\langle V(t), W(t)\rangle$ is constant. This shows that parallel transport preserves the scalar product.

(ii) For any smooth vector field W along γ we have

$$\frac{d}{dt}W(u(t)) = \dot{u}(t)\dot{W}(u(t))$$

and

$$\frac{\nabla}{dt}W(u(t)) = \dot{u}(t)\frac{\nabla W}{dt}(u(t)).$$

Thus, if W is parallel along γ, i.e. if $\frac{\nabla W}{dt} = 0$, then $\frac{\nabla}{dt}(W \circ u) = 0$, which means that $W \circ u$ is parallel along $\gamma \circ u$.

(iii) Let $V_1, \ldots, V_d$ be parallel vector fields along γ such that $(V_1(t), \ldots, V_d(t))$ is a basis for $T_{\gamma(t)}M$ for some (hence any) $t \in I$. Then

$$W(t) = \sum_i f^i(t)V_i(t)$$

for some smooth functions $f^i : I \to \mathbb{R}$, and

$$\frac{\nabla}{dt}W(t) = \sum_i \dot{f}^i(t)V_i(t).$$

On the other hand,

$$\mathrm{Hol}^{\gamma}_{t,t_0}(W(t)) = \sum_i f^i(t)V_i(t_0),$$

and

$$\left.\frac{d}{dt}\right|_{t_0} \mathrm{Hol}^{\gamma}_{t,t_0}(W(t)) = \sum_i \dot{f}^i(t_0)V_i(t_0) = \left.\frac{\nabla}{dt}\right|_{t_0} W(t). \qquad \square$$

One can express the Riemannian curvature tensor in terms of parallel transport along suitable curves, see [8, Theorem 12.47].

6.2 Geodesics

Let M be a manifold. A smooth curve $\gamma : I \to M$ is called a **geodesic** if its velocity field $\dot{\gamma}$ is parallel along γ, i.e. if for all $t \in I$ one has

$$\frac{\nabla}{dt}\dot{\gamma}(t) = 0,$$

or equivalently,

$$\ddot{\gamma}(t) \perp T_{\gamma(t)}M.$$

Example 6.2.1 Let $M = \mathbb{R}^n$. Then

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \ddot{\gamma}(t),$$

so γ is a geodesic if and only if it has the form

$$\gamma(t) = tv + w$$

for some $v, w \in \mathbb{R}^n$, which in turn means that γ parametrizes a straight line and has constant speed.

Example 6.2.2 Let $M = S^{n-1}$. Given vectors $p, v \in \mathbb{R}^n$ satisfying

$$\|p\| = \|v\| = 1, \quad \langle p, v\rangle = 0,$$

the curve

$$\gamma(t) := \cos t \cdot p + \sin t \cdot v$$

parametrizes a great circle on S^{n-1}. We have $\ddot{\gamma}(t) = -\gamma(t)$, so

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \Pi_{\gamma(t)}\ddot{\gamma}(t) = -\Pi_{\gamma(t)}\gamma(t) = 0,$$

hence γ is a geodesic.

Proposition 6.2.1 *Every geodesic has constant speed.*

Proof. If $\gamma : I \to M$ is a geodesic then

$$\frac{d}{dt}\|\dot{\gamma}(t)\|^2 = 2\left\langle \frac{\nabla}{dt}\dot{\gamma}(t), \dot{\gamma}(t)\right\rangle = 0. \qquad \square$$

Proposition 6.2.2 *Let I, J be open intervals, $u : J \to I$ a diffeomorphism, and $\gamma : I \to M$ a non-constant geodesic. Then the reparametrized curve $\gamma \circ u$ is a geodesic if and only if*

$$u(t) = at + b \tag{6.2}$$

for some constants $a, b \in \mathbb{R}$ with $a \neq 0$.

Proof. The first two ordinary derivatives of the curve $\gamma \circ u$ are

$$\frac{d}{dt}\gamma(u(t)) = \dot{u}(t) \cdot \dot{\gamma}(u(t)),$$
$$\left(\frac{d}{dt}\right)^2 \gamma(u(t)) = \ddot{u}(t) \cdot \dot{\gamma}(u(t)) + (\dot{u}(t))^2\, \ddot{\gamma}(u(t)).$$

Therefore,

$$\frac{\nabla}{dt}\frac{d}{dt}\gamma(u(t)) = \ddot{u}(t) \cdot \dot{\gamma}(u(t)).$$

Since $\dot{\gamma}$ never vanishes, the curve $\gamma \circ u$ is a geodesic if and only if $\ddot{u} \equiv 0$, which means that u has the form (6.2) with $a \neq 0$. $\square$

We will now express the geodesic equation in terms of a local parametrization, using our earlier calculation for parallel vector fields. Suppose M is a d–manifold, $\gamma : I \to M$ a smooth curve, and $F : U \to M$ a local parametrization whose image contains $\gamma(I)$. Let $\alpha := F^{-1} \circ \gamma$ have components $\alpha^1, \ldots, \alpha^d$. Applying (6.1) with $f^i = \dot{\alpha}^i$ we see that γ is a geodesic if and only if for all $t \in I$ one has

$$\ddot{\alpha}^k(t) + \sum_{ij} \tilde{\Gamma}^k_{ij}(\alpha(t))\dot{\alpha}^i(t)\dot{\alpha}^j(t) = 0, \quad k = 1, \ldots, d. \tag{6.3}$$

This is a system of second order, non-linear, ordinary differential equations. Therefore, for any $t_0 \in I$ and initial values $a \in U$, $b \in \mathbb{R}^d$ there is a unique solution α in some interval $(t_0 - \epsilon, t_0 + \epsilon)$ such that $\alpha(t_0) = a$ and $\dot{\alpha}(t_0) = b$, but we are not guaranteed a solution on the whole interval I.

Theorem 6.2.1 *Let $M \subset \mathbb{R}^n$ be a manifold.*

(i) *For any $p \in M$ and $v \in T_pM$ there exist an $\epsilon > 0$ and a geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.*

(ii) *For $i = 1, 2$ let $\gamma_i : I_i \to M$ be a geodesic. If there is a point $t_0 \in I_1 \cap I_2$ such that*

$$\gamma_1(t_0) = \gamma_2(t_0), \quad \dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$$

then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.

Proof. (i) This was proved above.

(ii) We first observe that the intersection $J := I_1 \cap I_2$ is a non-empty open interval. Let

$$A := \{t \in J \,|\, \gamma_1(t) = \gamma_2(t),\ \dot\gamma_1(t) = \dot\gamma_2(t)\}.$$

Because $\mathbb{R}^n$ is Hausdorff, the set A is closed in J. On the other hand, the local uniqueness of solutions to (6.3) with given inital values implies that A is open in J. Because J is connected and non-empty, we conclude that $A = J$. $\square$

Corollary 6.2.1 *Let M be a manifold and $p \in M$, $v \in T_pM$. Then there exists a unique maximal geodesic $\gamma : I \to M$ satisfying $0 \in I$, $\gamma(0) = p$, and $\dot\gamma(0) = v$.* $\square$

Here, "maximal" means that γ cannot be extended to a geodesic on a larger interval.

A manifold M is called **geodesically complete** if each of its maximal geodesics is defined on all of $\mathbb{R}$.

Proposition 6.2.3 *Let $\phi : M \to \bar{M}$ be a local isometry between manifolds and $\gamma : I \to M$ a smoooth curve. Then γ is a geodesic if and only if $\phi \circ \gamma$ is a geodesic.*

Proof. This follows from the geodesic equation (6.3) and the expresssion of the Christoffel symbols in terms of the first fundamental form given by Proposition 4.3.5 (compare the proof of Theorem 5.2.1). $\square$

6.3 Length-minimizing curves

Given a manifold M and two points $p, q \in M$, a smooth curve $\gamma : [a, b] \to M$ from p to q (i.e. satisfying $\gamma(a) = p$ and $\gamma(b) = q$) is called **length-minimizing** if for every smooth curve $\bar\gamma$ from p to q one has

$$L[\gamma] \le L[\bar\gamma],$$

where $L[\,\cdot\,]$ denotes the length of a curve as defined in Section 2.3. We are going to show that every length-minimizing curve of unit speed is a geodesic. To this end we consider one-parameter families of curves on M, given by a smooth map

$$h : I \times J \to M,$$

where I, J are intervals. For each $s \in I$ we have a curve

$$h_s : J \to M, \quad t \mapsto h(s,t).$$

Each partial derivative $\partial_j h$ is a vector field on M along h in the sense that

$$\partial_j h(s,t) \in T_{h(s,t)}M.$$

Let $\nabla_i \partial_j h$ be the covariant derivative of $\partial_j h$ with respect to the ith variable, i.e.

$$\nabla_i \partial_j h(s,t) := \Pi_{h(s,t)} \partial_i \partial_j h(s,t). \tag{6.4}$$

Lemma 6.3.1 *If $I, J \subset \mathbb{R}$ are intervals and $h : I \times J \to M$ a smooth map then*

$$\nabla_1 \partial_2 h = \nabla_2 \partial_1 h. \tag{6.5}$$

Proof. This follows from the definition (6.4) since $\partial_1 \partial_2 h = \partial_2 \partial_1 h$. □

Given a smooth curve $\gamma : [a,b] \to M$, a **variation** of γ is a smooth map $h : (-\epsilon, \epsilon) \times [a,b] \to M$ such that

$$h_0 = \gamma, \quad h(s,a) = \gamma(a), \quad h(s,b) = \gamma(b)$$

for all s. Here, the last two conditions mean that each curve h_s runs from p to q. The **variation field** of h is the vector field V along γ defined by

$$V(t) := \partial_1 h(0,t).$$

Proposition 6.3.1 *Let M be a manifold and $p, q \in M$. Suppose $\gamma : [a,b] \to M$ is a smooth curve from p to q and has unit speed. Let*

$$h : (-\epsilon, \epsilon) \times [a,b] \to M$$

be a variation of γ and V the corresponding variation field. Then

$$\left.\frac{d}{ds}\right|_0 L[h_s] = -\int_a^b \left\langle V(t), \frac{\nabla}{dt}\dot{\gamma}(t) \right\rangle dt.$$

Proof. We have

$$\begin{aligned}
\frac{d}{ds} L[h_s] &= \int_a^b \frac{d}{ds} \|\partial_2 h(s,t)\| \, dt \\
&= \int_a^b \|\partial_2 h(s,t)\|^{-1} \langle \nabla_1 \partial_2 h(s,t), \partial_2 h(s,t) \rangle \, dt \\
&= \int_a^b \|\partial_2 h(s,t)\|^{-1} \langle \nabla_2 \partial_1 h(s,t), \partial_2 h(s,t) \rangle \, dt.
\end{aligned}$$

At $s = 0$ this becomes

$$\left.\frac{d}{ds}\right|_0 L[h_s] = \int_a^b \left\langle \frac{\nabla}{dt} V(t), \dot{\gamma}(t) \right\rangle dt. \tag{6.6}$$

Observing that $V(a) = 0$ and $V(b) = 0$, we get

$$\begin{aligned} 0 &= \langle V(b), \dot{\gamma}(b) \rangle - \langle V(a), \dot{\gamma}(a) \rangle \\ &= \int_a^b \frac{d}{dt} \langle V(t), \dot{\gamma}(t) \rangle dt \\ &= \int_a^b \left\langle \frac{\nabla}{dt} V(t), \dot{\gamma}(t) \right\rangle dt + \int_a^b \left\langle V(t), \frac{\nabla}{dt} \dot{\gamma}(t) \right\rangle dt. \end{aligned}$$

Combining this with (6.6) we obtain the proposition. □

Before applying the last proposition to length-minimizing curves, we will prove a lemma which is useful for constructing variations of a given curve.

Given sets $A \subset B$, a map $R : B \to A$ is called a **retraction** if $R(x) = x$ for all $x \in A$.

Lemma 6.3.2 *If $M \subset \mathbb{R}^n$ is a manifold and $p \in M$ then there exist a neighbourhood W of p in $\mathbb{R}^n$ and a smooth retraction $R : W \to M \cap W$.*

Proof. Let $(T_pM)^\perp$ be the orthogonal complement of the tangent space T_pM in $\mathbb{R}^n$. We define

$$\phi : M \times (T_pM)^\perp \to \mathbb{R}^n, \quad (x, w) \mapsto x + w.$$

The differential

$$d\phi_{(p,0)} : T_pM \times (T_pM)^\perp \to \mathbb{R}^n$$

is given by

$$d\phi_{(p,0)}(v, w) = v + w,$$

hence $d\phi_{(p,0)}$ is an isomorphism. For any $r > 0$ let $B_r \subset (T_pM)^\perp$ be the open r–ball centred at the origin. By the inverse function theorem, Theorem 1.4.1, there exist a neighbourhood N_0 of p in M and $r_0 > 0$ such that ϕ maps $N_0 \times B_{r_0}$ diffeomorphically onto an open set $W_0 \subset \mathbb{R}^n$. Choose an open set $W_1 \subset \mathbb{R}^n$ such that $M \cap W_1 = N_0$. By continuity of ϕ we can find a neighbourhood N of p in N_0 and $r \in (0, r_0]$ such that

$$\phi(N \times B_r) \subset W_1.$$

Let $W := \phi(N \times B_r)$. Because ϕ is injective on $N_0 \times B_{r_0}$ we have $M \cap W = N$. Let

$$\psi : W \to N \times B_r$$

be the inverse of $\phi|_{N \times B_r}$. We now define $R : W \to N$ to be the composition of ψ with the projection $N \times B_r \to N$. $\square$

Theorem 6.3.1 *Let M be a manifold and $p, q \in M$. Suppose $\gamma : [a, b] \to M$ is a length-minimizing smooth curve from p to q and has unit speed. Then γ is a geodesic.*

Proof. If h is any variation of γ then the function $L[h_t]$ has a minimum at $t = 0$, so by Proposition 6.3.1 we have

$$\int_a^b \left\langle V(t), \frac{\nabla}{dt}\dot{\gamma}(t) \right\rangle dt = 0.$$

We will deduce from this that $\frac{\nabla}{dt}\dot{\gamma}(t) = 0$ by constructing suitable variations h.

Since $\frac{\nabla}{dt}\dot{\gamma}$ is continuous, it suffices to show that it vanishes at every interior point $t_0 \in (a, b)$. By Lemma 6.3.2 we can find a neighbourhood W of $\gamma(t_0)$ in $\mathbb{R}^n$ and a smooth retraction $R : W \to M \cap W$. Choose $\epsilon, \delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset [a, b]$ and

$$\gamma(t) + s\frac{\nabla}{dt}\dot{\gamma}(t) \in W \quad \text{when } |t - t_0| < \delta,\ |s| < \epsilon.$$

Choose a smooth function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(t_0) > 0$, and

$$\beta(t) = 0 \quad \text{when } |t - t_0| \geq \delta/2,$$

see Problem 6.1. Define

$$h : (-\epsilon, \epsilon) \times [a, b] \to M$$

by

$$h(s, t) := \begin{cases} R\left(\gamma(t) + s\beta(t)\frac{\nabla}{dt}\dot{\gamma}(t)\right) & \text{when } |t - t_0| < \delta, \\ \gamma(t) & \text{else.} \end{cases}$$

Note that $h(s, t) = \gamma(t)$ when $|t - t_0| \geq \delta/2$, and this ensures that h is smooth. Because $dR_z(v) = v$ for any $z \in M \cap W$ and $v \in T_zM$, the variation field of h is

$$V(t) = \beta(t)\frac{\nabla}{dt}\dot{\gamma}(t).$$

As explained above, this implies that

$$\int_a^b \beta(t) \left\| \frac{\nabla}{dt} \dot{\gamma}(t) \right\|^2 dt = 0,$$

hence $\frac{\nabla}{dt}\dot{\gamma}$ vanishes at t_0. $\square$

6.4 Geodesics on surfaces of revolution

In this section we will prove Clairaut's theorem, which gives a simple description of the geodesics on a surface of revolution. Our approach is influenced by that in [1].

As in Section 3.4, let $C \subset \mathbb{R}^2$ be a regular curve contained in the right half-plane, and let $S \subset \mathbb{R}^3$ be the corresponding surface of revolution. Let $\gamma : I \to C$ be a local parametrization of C and $\gamma(t) = (r(t), h(t))$. Let $F : I \times J \to S$ be a local parametrization of S as in (3.2).

Proposition 6.4.1 *Let S, F be as above and $(t_0, \theta_0) \in I \times J$.*

(i) *The curve $t \mapsto F(t, \theta_0)$ is a geodesic if and only if γ has constant speed.*

(ii) *The curve $\theta \mapsto F(t_0, \theta)$ is a geodesic if and only if $\dot{r}(t_0) = 0$.*

Proof. We prove (ii) and leave (i) for the reader. Let $c(\theta) := F(t_0, \theta)$. Since $\{\partial_i F(t, \theta)\}_{i=1,2}$ is a basis for the tangent space $T_{F(t,\theta)}S$ for every (s, t), it follows that c is a geodesic if and only if

$$\langle \ddot{c}(\theta), \partial_i F(t_0, \theta) \rangle = 0, \quad i = 1, 2.$$

We have

$$\ddot{c}(\theta) = (-r(t)\cos\theta, -r(t)\sin\theta, 0),$$

and using (3.3) we find

$$\begin{aligned} \langle \ddot{c}(t), \partial_1 F(t_0, \theta) \rangle &= -r(t_0)\dot{r}(t_0), \\ \langle \ddot{c}(t), \partial_2 F(t_0, \theta) \rangle &= 0. \end{aligned}$$

Recalling the assumption $r > 0$, we conclude that c is a geodesic if and only if $\dot{r}(t_0) = 0$. $\square$

Choose an orientation of the regular curve C. From now on we will always assume that the local parametrization $\gamma : I \to C$ is orientation preserving and has unit speed. There are smooth vector fields X_1, X_2 on S

such that if F is any local parametrization of S as above then the restrictions of X_1, X_2 to $F(U)$ are the coordinate vector fields of F. For $p = (p^1, p^2, p^3) \in S$ one has

$$X_2(p) = (-p^2, p^1, 0).$$

To describe $X_1(p)$, recall from Lemma 3.4.1 that we can uniquely represent p in the form $p = (ax, bx, z)$ where $(x, z) \in C$ and $(a, b) \in S^1$. If $(u, v) \in T_{(x,z)}C$ is the positively oriented tangent vector of unit length then

$$X_1(p) = (au, bu, v).$$

The following lemma is an expression of the rotational symmetry of S.

Lemma 6.4.1 *For every smooth vector field Y on S one has*

$$\langle \nabla_Y X_2, Y \rangle = 0.$$

Proof. The statement is local, so it suffices to verify it in the image of a local parametrization F as above. Recall from (3.4) that the coefficients $\tilde{g}_{ij}$ of the first fundamental form of S relative to F are independent of θ. This means that $\partial_{X_2} g_{ij} = 0$. Recalling that $\nabla_{X_i} X_j = \nabla_{X_j} X_i$ by Proposition 4.3.4 we obtain

$$\begin{aligned} 0 &= \partial_{X_2} \langle X_i, X_j \rangle \\ &= \langle \nabla_{X_2} X_i, X_j \rangle + \langle X_i, \nabla_{X_2} X_j \rangle \\ &= \langle \nabla_{X_i} X_2, X_j \rangle + \langle X_i, \nabla_{X_j} X_2 \rangle. \end{aligned}$$

For $i = j$ this yields

$$\langle \nabla_{X_i} X_2, X_i \rangle = 0.$$

On the image of F we can write

$$Y = \sum_i a^i X_i$$

for some smooth real-valued functions a^i. Now

$$\begin{aligned} \langle \nabla_Y X_2, Y \rangle &= \sum_{ij} a^i a^j \langle \nabla_{X_i} X_2, X_j \rangle \\ &= \sum_{i<j} a^i a^j \left(\langle \nabla_{X_i} X_2, X_j \rangle + \langle \nabla_{X_j} X_2, X_i \rangle \right) \\ &= 0. \qquad \square \end{aligned}$$

Corollary 6.4.1 *For every $p \in S$ and $v \in T_pS$ one has*

$$\langle \nabla_{p,v} X_2, v \rangle = 0.$$

Proof. Choose a smooth vector field Y on S such that $Y_p = v$ (see Example 4.3.1). Then $\langle \nabla_{p,v} X_2, v \rangle$ is the value of $\langle \nabla_Y X_2, Y \rangle$ at p, hence equals zero. □

The **angle** between two non-zero vectors $v, w \in \mathbb{R}^n$ is defined by

$$\angle(v, w) := \arccos \frac{\langle v, w \rangle}{\|v\| \|w\|}.$$

Theorem 6.4.1 *Let S be a surface of revolution as above and $\alpha : K \to S$ a regular parametrized curve, where K is an open interval. For any $s \in K$ let $\phi(s)$ be the angle between the vectors $\dot{\alpha}(s)$ and $X_2(\alpha(s))$ (the latter being tangent to the parallel through $\alpha(s)$), and let $\rho(s)$ denote the distance from $\alpha(s)$ to the z–axis.*

(i) *If α is a geodesic then there is a constant A such that*

$$\rho(s) \cdot \cos \phi(s) = A \quad \textit{for all } s \in K. \tag{6.7}$$

(ii) *If α has constant speed and (6.7) holds then either α is a geodesic or it takes values on a parallel of S.*

Explicitly, if α has components $\alpha^1, \alpha^2, \alpha^3$ then

$$\rho(s) = \sqrt{(\alpha^1(s))^2 + (\alpha^2(s))^2}.$$

Proof. We first apply Corollary 6.4.1 to derive a condition equivalent to (6.7) under the assumption that α has constant speed (which always holds if α is a geodesic). We have

$$\begin{aligned} \langle \dot{\alpha}(s), X_2(\alpha(s)) \rangle &= \cos \phi(s) \cdot \|\dot{\alpha}(s)\| \cdot \|X_2(\alpha(s))\| \\ &= \rho(s) \cdot \cos \phi(s) \cdot \|\dot{\alpha}(s)\|. \end{aligned}$$

If α has constant, positive speed then

$$\rho(s) \cdot \cos \phi(s) \text{ constant} \iff \langle \dot{\alpha}(s), X_2(\alpha(s)) \rangle \text{ constant}.$$

But

$$\begin{aligned} \frac{d}{ds} \langle \dot{\alpha}(s), X_2(\alpha(s)) \rangle &= \left\langle \frac{\nabla}{ds} \dot{\alpha}(s), X_2(\alpha(s)) \right\rangle + \left\langle \dot{\alpha}(s), \frac{\nabla}{ds} X_2(\alpha(s)) \right\rangle \\ &= \left\langle \frac{\nabla}{ds} \dot{\alpha}(s), X_2(\alpha(s)) \right\rangle, \end{aligned}$$

since, by Corollary 6.4.1,

$$\left\langle \frac{\nabla}{ds} X_2(\alpha(s)), \dot{\alpha}(s) \right\rangle = \langle \nabla_{\alpha(s),\dot{\alpha}(s)} X_2, \dot{\alpha}(s) \rangle = 0.$$

Thus,

$$\rho(s) \cdot \cos\phi(s) \text{ constant} \Longleftrightarrow \left\langle X_2(\alpha(s)), \frac{\nabla}{ds}\dot{\alpha}(s) \right\rangle = 0 \text{ for all } s. \tag{6.8}$$

This proves (i). To prove (ii), suppose α has constant speed and (6.7) holds. Then

$$0 = \frac{d}{ds}\|\dot{\alpha}(s)\|^2 = 2\left\langle \frac{\nabla}{ds}\dot{\alpha}(s), \dot{\alpha}(s) \right\rangle.$$

Now suppose in addition that α is not a geodesic, i.e.

$$\frac{\nabla \dot{\alpha}}{ds}(s_0) \neq 0$$

for some $s_0 \in K$. Let $L \subset K$ be the largest open subinterval containing s_0 such that $\frac{\nabla}{ds}\dot{\alpha}(s) \neq 0$ for every $s \in L$. Because S has dimension 2, it follows that $\dot{\alpha}(s)$ must be a real multiple of $X_2(\alpha(s))$ for every s, i.e. there is a nowhere vanishing function $b : L \to \mathbb{R}$ such that

$$\begin{aligned} \dot{\alpha}^1 &= -b\alpha^2, \\ \dot{\alpha}^2 &= b\alpha^1, \\ \dot{\alpha}^3 &= 0. \end{aligned} \tag{6.9}$$

In terms of the curve

$$\beta(s) := \alpha^1(s) + i\alpha^2(s)$$

in the complex plane, the first two equations in (6.9) can be rephrased as

$$\dot{\beta} = ib\beta.$$

If $B : L \to \mathbb{R}$ is a primitive of b (i.e. $\dot{B} = b$) then we must have

$$\beta(s) = \beta_0 e^{iB(s)}$$

for some complex number β_0. Then

$$\dot{\beta}(s) = i\beta_0 b(s) e^{iB(s)},$$

so

$$0 < \text{const} = \|\dot{\alpha}(s)\| = |\dot{\beta}(s)| = |\beta_0| \cdot |b(s)|.$$

Therefore, $|b(s)|$ is constant and non-zero. Since b is continuous, it must be constant, say $b \equiv b_0$. After expressing β_0 in polar coordinates we see that there are real numbers r_0, s_1, where $r_0 \neq 0$, such that

$$\beta(s) = r_0 e^{ib_0(s+s_1)}.$$

After perhaps replacing s_1 by $s_1 + \pi/b_0$ we may assume that $r_0 > 0$. We conclude that there is a real number h_0 such that

$$\alpha(s) = (r_0 \cos(b_0(s+s_1)), r_0 \sin(b_0(s+s_1)), h_0) \tag{6.10}$$

for all $s \in L$. Since α takes values on S, the point (r_0, h_0) must lie on the regular curve C, and $\alpha(L)$ is contained in a parallel of S. To complete the proof we only need to show that $L = K$. Suppose $L \neq K$. Then L must have an end-point s' which lies in K.

Choose an orientation preserving local parametrization $\gamma : I \to C$ of unit speed such that $\gamma(t_0) = (r_0, h_0)$ for some $t_0 \in I$. The equation (6.10) shows that $\alpha|_L$ is a linear reparametrization of the kind of parametrized curve considered in Proposition 6.4.1 (i). As in the proof of that lemma we find that for $s \in L$,

$$\begin{aligned}\langle \ddot{\alpha}(s), X_1(\alpha(s))\rangle &= -b^2 r(t_0)\dot{r}(t_0),\\ \langle \ddot{\alpha}(s), X_2(\alpha(s))\rangle &= 0.\end{aligned}$$

By assumption, $\frac{\nabla}{ds}\dot{\alpha}(s) \neq 0$ for all $s \in L$, hence $\dot{r}(t_0) \neq 0$. Because the continuous function

$$K \to \mathbb{R}, \quad s \mapsto \langle \ddot{\alpha}(s), X_1(\alpha(s))\rangle$$

is non-zero and constant on L, it follows that it is also non-zero at the endpoint s'. But then $\frac{\nabla}{ds}\dot{\alpha}(s) \neq 0$ for $s = s'$ and therefore for all s in an open interval L' containing s'. Since $L \cup L'$ is an open interval, we have a contradiction. This shows that $L = K$ and the proof is complete. $\square$

6.5 The exponential map

Let M be a manifold and $p \in M$. If $\gamma : I \to M$ is a parametrized curve such that $0 \in I$ and $\gamma(0) = p$, then we say that γ **starts at** p. We will now collect all the geodesics starting at p into a single map $\exp_p$, which will be used to construct a special kind of local parametrization.

For each $v \in T_pM$ let γ_v be the maximal geodesic starting at p such that $\dot{\gamma}(0) = v$. For $s \in \mathbb{R}$ one has

$$\frac{d}{dt}\gamma_v(st) = s\dot{\gamma}_v(st), \qquad \left.\frac{d}{dt}\right|_0 \gamma_v(st) = sv.$$

We now apply the uniqueness result for geodesics, Theorem 6.2.1, to conclude that $\gamma_v(st)$ is defined if and only if $\gamma_{sv}(t)$ is defined, and in that case

$$\gamma_{sv}(t) = \gamma_v(st).$$

Let

$$\mathcal{D}_p := \{v \in T_pM \mid \gamma_v(1) \text{ is defined}\}.$$

The map

$$\exp_p : \mathcal{D}_p \to M, \quad v \mapsto \gamma_v(1)$$

is called the **exponential map** at p. We observe that if $\exp_p(tv)$ is defined then

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t). \tag{6.11}$$

Proposition 6.5.1 *For any $p \in M$ the following hold*

(i) *If $v \in \mathcal{D}_p$ then $sv \in \mathcal{D}_p$ for $0 \le s \le 1$.*

(ii) *$\mathcal{D}_p$ is an open subset of T_pM.*

(iii) *$\exp_p$ is smooth.*

Proof. Parts (ii) and (iii) follow from the theory of ordinary differential equations. To prove (i), suppose $v \in \mathcal{D}_p$ and $0 \le s \le 1$. Then γ_v is defined on an open interval containing 0 and 1, hence $\gamma_v(s)$ is defined. As we saw above, this implies that $\gamma_{sv}(1)$ is defined, so $sv \in \mathcal{D}_v$. $\square$

Example 6.5.1 Let $M = \mathbb{R}^n$ and $p \in \mathbb{R}^n$. For every $v \in T_p\mathbb{R}^n = \mathbb{R}^n$, the geodesic γ_v is defined on all of $\mathbb{R}$, and by Example 6.2.1 we have

$$\gamma_v(t) = p + tv.$$

Therefore, $\mathcal{D}_p = \mathbb{R}^n$ and

$$\exp_p(v) = p + v.$$

Example 6.5.2 Let $M = S^{n-1}$. As we saw in Example 6.2.2, every maximal geodesic in S^{n-1} is defined on all of $\mathbb{R}$, hence

$$\mathcal{D}_p = T_pS^{n-1} = p^\perp$$

for every $p \in S^{n-1}$. Moreover, if $w \in T_pS^{n-1}$ and $\|w\| = 1$ then

$$\exp_p(tw) = \cos t \cdot p + \sin t \cdot w.$$

We will now determine the differential of the exponential map $\exp_p$ at the origin. Since $\mathcal{D}_p$ is an open subset of T_pM, we have

$$T_0\mathcal{D}_p = T_0(T_pM) = T_pM,$$

hence the differential $d(exp_p)_0$ is a linear map $T_pM \to T_pM$.

Proposition 6.5.2 *The differential of* $\exp_p : \mathcal{D}_p \to M$ *at the origin is the identity map.*

Proof. Using (6.11) we find that for every $v \in T_pM$,

$$d(exp_p)_0(v) = \left.\frac{d}{dt}\right|_0 \exp_p(tv) = v. \qquad \square$$

The inverse function theorem, Theorem 1.4.1, now says that $\exp_p$ maps some neighbourhood of 0 in T_pM diffeomorphically onto an open subset of M. This yields the following result.

Proposition 6.5.3 *Let M be a d–manifold and $p \in M$. If $\{v_i\}$ is an orthonormal basis for T_pM then there exists a neighbourhood U of 0 in $\mathbb{R}^d$ such that the map*

$$F : U \to M, \quad (x^1, \ldots, x^d) \mapsto \exp_p\left(\sum_{i=1}^{d} x^i v_i\right)$$

is a local parametrization. $\square$

If $p = F(x^1, \ldots, x^d)$ then $x^1, \ldots, x^d$ are called **normal coordinates** for p.

6.6 Geodesic polar coordinates

Whereas in the previous section we studied manifolds of arbitrary dimension, we will now focus on 2–manifolds $S \subset \mathbb{R}^n$. We will combine the exponential map at a given point $p \in S$ with polar coordinates in the tangent space T_pS to obtain **geodesic polar coordinates** on S. These will be used to show

that regular surfaces with the same constant Gauss curvature are locally isometric.

For $\epsilon > 0$ let $B_{p,\epsilon}$ be the open ϵ–ball in T_pS centred at the origin. Choose ϵ so small that $\exp_p$ maps $B_{p,\epsilon}$ diffeomorphically onto an open set in S. Let (v_1, v_2) be an orthonormal basis for T_pS and define

$$h : (-\epsilon, \epsilon) \times \mathbb{R} \to S, \quad (r, \theta) \mapsto \exp_p(r \cos\theta \cdot v_1 + r \sin\theta \cdot v_2).$$

Note that for any real number η, the restriction $F := h|_{(0,\epsilon)\times(\eta,\eta+2\pi)}$ is a local parametrization of S.

Lemma 6.6.1 (Gauss) *If F is a local parametrization of S as above then the matrix $(\tilde{g}_{ij})$ of the first fundamental form on S has the form*

$$(\tilde{g}_{ij}(r, \theta)) = \begin{pmatrix} 1 & 0 \\ 0 & f(r, \theta)^2 \end{pmatrix},$$

where $f(r, \theta) > 0$ and for all θ,

$$\lim_{r \to 0^+} \frac{f(r, \theta)}{r} = 1.$$

Proof. Let

$$c(\theta) := \cos\theta \cdot v_1 + \sin\theta \cdot v_2,$$

so that

$$h(r, \theta) = \exp_p(rc(\theta)).$$

To compute $\tilde{g}_{11} = \|\partial h/\partial r\|^2$, we first observe that for each θ the curve $r \mapsto h(r, \theta)$ is a geodesic and therefore has constant speed. Since

$$\frac{\partial h}{\partial r}(0, \theta) = c(\theta), \quad \|c(\theta)\| = 1,$$

we have

$$\left\| \frac{\partial h}{\partial r}(r, \theta) \right\| = 1$$

for all r, hence $\tilde{g}_{11} = 1$.

To prove the assertion about $\tilde{g}_{22} = \|\partial h/\partial \theta\|^2$, we first compute

$$\frac{\partial h}{\partial \theta} = d(\exp_p)_{rc(\theta)}(r\dot{c}(\theta)) = r \cdot d(\exp_p)_{rc(\theta)}(\dot{c}(\theta)).$$

This yields

$$\frac{1}{r}\frac{\partial h}{\partial \theta} \to d(\exp_p)_0(\dot{c}(\theta)) = \dot{c}(\theta) \quad \text{as } r \to 0.$$

Because $\|\dot{c}(\theta)\| = 1$,

$$\frac{1}{r}\left\|\frac{\partial h}{\partial \theta}\right\| \to 1 \quad \text{as } r \to 0^+.$$

Defining $f := \|\partial h/\partial\theta\|$, this proves the assertion about $\tilde{g}_{22} = f^2$.

We will compute $\tilde{g}_{12} = \langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\rangle$ in two steps. We first show that $\langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\rangle$ does not depend on r, and then observe that it vanishes for $r = 0$. We begin by noting that

$$\frac{\nabla}{dr}\frac{\partial h}{\partial r} = 0,$$

since the curve $r \mapsto h(r,\theta)$ is a geodesic. Combining this with Lemma 6.3.1 we obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\left\langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right\rangle &= \left\langle \frac{\nabla}{dr}\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right\rangle + \left\langle \frac{\partial h}{\partial r}, \frac{\nabla}{dr}\frac{\partial h}{\partial \theta}\right\rangle \\
&= \left\langle \frac{\partial h}{\partial r}, \frac{\nabla}{d\theta}\frac{\partial h}{\partial r}\right\rangle \\
&= \frac{1}{2}\frac{\partial}{\partial \theta}\left\|\frac{\partial h}{\partial r}\right\|^2 \\
&= 0.
\end{aligned}$$

Therefore, $\langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\rangle$ is constant in r. Since $h(0,\theta) = p$ for all θ, we have

$$\frac{\partial h}{\partial \theta}(0,\theta) = 0,$$

hence $\langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\rangle$ vanishes for $r = 0$. $\square$

Lemma 6.6.2 *Suppose a regular surface S has a local parametrization $F(r,\theta)$ such that*

$$(\tilde{g}_{ij}(r,\theta)) = \begin{pmatrix} 1 & 0 \\ 0 & f(r,\theta)^2 \end{pmatrix}$$

for some function $f(r,\theta)$. Then the Gauss curvature K of S satisfies

$$K(F(r,\theta)) = -\frac{1}{f(r,\theta)} \cdot \frac{\partial^2 f}{\partial r^2}(r,\theta).$$

In short, with $\tilde{K} = K \circ F$,

$$\tilde{K} = -\frac{1}{f} \cdot \frac{\partial^2 f}{\partial r^2}.$$

Proof. Let (X_1, X_2) be the coordinate frame of the local parametrization $F : U \to S$. By assumption, the matrix of functions $g_{ij} = \langle X_i, X_j \rangle$ has the form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2 \end{pmatrix},$$

where $\phi = \|X_2\|$. Note that $f = \phi \circ F$. On $F(U)$ we therefore have an orthonormal frame $(X_1, \phi^{-1}X_2)$, and Theorem 5.3.2 says that

$$\begin{aligned} K &= -\langle R(X_1, \phi^{-1}X_2)X_1, \phi^{-1}X_2 \rangle \\ &= -\phi^{-2}\langle R(X_1, X_2)X_1, X_2 \rangle \\ &= -\phi^{-2}\langle \textstyle\sum_\ell R^\ell_{121} X_\ell, X_2 \rangle \\ &= -\phi^{-2}\|X_2\|^2 R^2_{121} \\ &= -R^2_{121}. \end{aligned}$$

To compute R^2_{121} we will first compute the Christoffel symbols Γ^k_{ij} using Proposition 4.3.5 and then apply Proposition 5.2.4. To simplify notation we write ∂_i instead of ∂_{X_i}.

The inverse of the matrix (g_{ij}) is

$$(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^{-2} \end{pmatrix}.$$

Proposition 4.3.5 yields

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}\sum_\ell g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \\ &= \frac{1}{2} g^{kk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \end{aligned}$$

Since g_{ij} is constant unless $(i, j) = (2, 2)$, we have

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = 0.$$

Furthermore,

$$\Gamma^2_{12} = \phi^{-1} \cdot \partial_1\phi, \quad \Gamma^1_{22} = -\phi \cdot \partial_1\phi, \quad \Gamma^2_{22} = \phi^{-1} \cdot \partial_2\phi.$$

By Proposition 5.2.4,

$$\begin{aligned} R^2_{121} &= \partial_1\Gamma^2_{12} - \partial_2\Gamma^2_{11} + \sum_m \left(\Gamma^m_{12}\Gamma^2_{1m} - \Gamma^m_{11}\Gamma^2_{2m}\right) \\ &= \partial_1\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} \\ &= \partial_1(\phi^{-1} \cdot \partial_1\phi) + (\phi^{-1} \cdot \partial_1\phi)^2 \\ &= \phi^{-1}\partial_1\partial_1\phi. \end{aligned}$$

Altogether, we therefore have

$$K = -R^2_{121} = -\phi^{-1}\partial_1\partial_1\phi,$$

which is equivalent to the statement of the proposition. □

Proposition 6.6.1 *In the situation of Lemma 6.6.1, if S has constant Gauss curvature K then*

$$f(r,\theta) = \begin{cases} \frac{1}{\omega}\sin(\omega r) & \text{if } K \equiv \omega^2,\ \omega > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\omega}\sinh(\omega r) & \text{if } K \equiv -\omega^2,\ \omega > 0. \end{cases}$$

Proof. By Lemmas 6.6.1 and 6.6.2 we know that $\tilde{K}$ is given by

$$\tilde{K} = -\frac{1}{f}\frac{\partial^2 f}{\partial r^2},$$

where the function f satisfies

$$\frac{1}{r}f(r,\theta) \to 1 \quad \text{as } r \to 0^+. \tag{6.12}$$

We now distinguish between three cases.

(i) If $K = \omega^2$, $\omega > 0$, then

$$\frac{\partial^2 f}{\partial r^2} = -\omega^2 f.$$

From the theory of ordinary linear differential equations we know that

$$f(r,\theta) = a(\theta)\cos(\omega r) + b(\theta)\sin(\omega r)$$

for some real-valued functions a, b. Because

$$\cos 0 = 1, \quad \lim_{r\to 0}\frac{\sin(\omega r)}{r} = \omega,$$

the condition (6.12) yields

$$a = 0, \quad b = \frac{1}{\omega},$$

hence $f(r,\theta) = \frac{1}{\omega}\sin(\omega r)$.

(ii) If $K = 0$, then

$$\frac{\partial^2 f}{\partial r^2} = 0,$$

hence

$$f(r, \theta) = a(\theta)r + b(\theta).$$

The condition 6.12 gives $a = 1$, $b = 0$, so that $f(r, \theta) = r$.

(iii) $K = -\omega^2$, $\omega > 0$, then

$$\frac{\partial^2 f}{\partial r^2} = \omega^2 f.$$

We now have

$$f(r, \theta) = a(\theta)\cosh(\omega r) + b(\theta)\sinh(\omega r),$$

and as in Case (i) we find that $a = 0$, $b = \frac{1}{\omega}$. This gives $f(r, \theta) = \frac{1}{\omega}\sinh(\omega r)$. $\square$

Theorem 6.6.1 *Let $S, \bar{S}$ be regular surfaces with the same constant Gauss curvature. Then for any $p \in S$, $\bar{p} \in \bar{S}$, and linear isometry $A : T_pS \to T_{\bar{p}}\bar{S}$ there exist neighbourhoods $V \subset S$ of p and $\bar{V} \subset \bar{S}$ of $\bar{p}$ and an isometry $\phi : V \to \bar{V}$ such that $\phi(p) = \bar{p}$ and $d\phi_p = A$.*

Proof. The definition of ϕ is forced upon us by the naturality of the exponential map, see Problem 6.6. Choose $\epsilon > 0$ such that

- $\exp_p$ maps the ball $B_{p,\epsilon}$ in T_pM diffeomorphically onto an open set $V \subset S$, and
- $\exp_{\bar{p}}$ maps the ball $B_{\bar{p},\epsilon}$ in $T_{\bar{p}}\bar{M}$ diffeomorphically onto an open set $\bar{V} \subset \bar{S}$.

Then the map

$$\phi := \exp_{\bar{p}} \circ A \circ \exp_p^{-1} : V \to \bar{V}$$

is a diffeomorphism. We will use geodesic polar coordinates to show that ϕ is an isometry. Choose an orthonormal basis (ξ_1, ξ_2) for T_pS and define $\bar{\xi}_i := A\xi_i$. Then $(\bar{\xi}_1, \bar{\xi}_2)$ is an orthonormal basis for $T_{\bar{p}}\bar{S}$. For any real number η we have local parametrizations $F, \bar{F}$ of $S, \bar{S}$ respectively, both defined on $(0, \epsilon) \times (\eta, \eta + 2\pi)$, given by

$$F(r, \theta) = \exp_p(r\cos\theta \cdot \xi_1 + r\sin\theta \cdot \xi_2),$$
$$\bar{F}(r, \theta) = \exp_{\bar{p}}(r\cos\theta \cdot \bar{\xi}_1 + r\sin\theta \cdot \bar{\xi}_2).$$

By definition of ϕ we have

$$\begin{aligned}\phi(F(r,\theta)) &= \phi(\exp_p((r\cos\theta\cdot\xi_1 + r\sin\theta\cdot\xi_2))) \\ &= \exp_{\bar{p}}(A(r\cos\theta\cdot\xi_1 + r\sin\theta\cdot\xi_2)) \\ &= \bar{F}(r,\theta).\end{aligned}$$

We now invoke the assumption that S and $\bar{S}$ have the same constant Gauss curvature. By Lemma 6.6.1 and Proposition 6.6.1 this implies that the components $\tilde{g}_{ij}$ of the first fundamental form of S with respect to F is the same as for $\bar{S}$ with respect to $\bar{F}$. Explicitly,

$$\langle \partial_i F, \partial_j F\rangle = \langle \partial_i \bar{F}, \partial_j \bar{F}\rangle$$

for all i, j. This shows that $d\phi_q$ is a linear isometry for every q in the image of F. By varying η we conclude that $d\phi_q$ is a linear isometry for every $q \in V\backslash\{p\}$. But $d\phi_p = A$ by Proposition 6.5.2, hence ϕ is an isometry. $\square$

Problems

6.1

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Show that f is smooth.

(b) Given $a \in \mathbb{R}$ and $\delta > 0$, construct a smooth function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(a) > 0$, and $\beta(x) = 0$ when $|x - a| \ge \delta$.

6.2 Let $\phi : M \to \bar{M}$ be a local diffeomorphism and $\gamma : I \to M$ a smooth curve. Let $\bar{W}$ be a smooth vector field on $\bar{M}$ along the curve $\bar{\gamma} := \phi \circ \gamma : I \to \bar{M}$. Prove the following.

(a) There is a unique smooth vector field $W = \phi^*\bar{W}$ on M along γ such that for all $t \in I$,

$$d\phi_{\gamma(t)}(W(t)) = \bar{W}(t).$$

(Compare Problem 1.17.)

(b) If ϕ is a local isometry, then W is parallel if and only if $\bar{W}$ is parallel.

6.3 Let C be a circle on the unit sphere $S^2 \subset \mathbb{R}^3$, i.e. a non-trivial intersection of S^2 with an affine plane. Compute the parallel transport once around C.

Hint: If C is not a great circle, consider the cone V tangent to S^2 along C. Show that the parallel transport along C is the same whether taken relative to S^2 or V.

6.4 Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth curve. Show that

$$L[\gamma] \geq \|\gamma(b) - \gamma(a)\|.$$

6.5 Prove Proposition 6.4.1 (i).

6.6 Let $\phi : M \to \bar{M}$ be a local isometry and $p \in M$, $v \in T_pM$. Show that if $\exp_p(v)$ is defined, then $\exp_{\phi(p)}(d\phi_p(v))$ is also defined, and

$$\phi(\exp_p(v) = \exp_{\phi(p)}(d\phi_p(v)).$$

Hint: Use Proposition 6.2.3.

Chapter 7

Differential forms

This section will provide some background on differential forms required for the proof of the Gauss-Bonnet theorem to be given in Section 8. For a thorough discussion of differential forms we refer to [14, 9].

7.1 Alternating tensors

Let V be a finite-dimensional real vector space and $V^k := V \times \cdots \times V$ the k–fold Cartesian product, where k is a positive integer. A multilinear map

$$f : V^k \to \mathbb{R}$$

is called a k–**tensor** on V. A k–tensor f is **alternating** if for every permutation σ of the set $\{1, \ldots, k\}$ and all $v_1, \ldots, v_k \in V$ one has

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \ldots, v_k),$$

where $\operatorname{sgn}(\sigma) = \pm 1$ is the sign of the permutation. Let $A_k(V)$ be the real vector space consisting of all alternating k–tensors on V. A 0–tensor on V is by definition just a real number, and we define $A_0(V) := \mathbb{R}$.

Note that $A_1(V)$ is the **dual space** V^* consisting of all linear maps $V \to \mathbb{R}$, whereas $A_2(V)$ is the space of all skew-symmetric bilinear maps $V \times V \to \mathbb{R}$.

Lemma 7.1.1 *If V has dimension d then $A_d(V)$ has dimension* 1.

Proof. Suppose $f \in A_d(V)$, and let $(e_1, \ldots, e_d)$ and $(v_1, \ldots, v_d)$ be two d–tuples of elements of V such that

$$v_j = \sum_i b_{ij} e_i$$

for some matrix $B = (b_{ij})$. Then

$$\begin{aligned} f(v_1, \ldots, v_d) &= \sum_\sigma \operatorname{sgn}(\sigma) b_{\sigma(1),1} \cdots b_{\sigma(d),d} f(e_1, \ldots, e_d) \\ &= \det(B) \cdot f(e_1, \ldots, e_d), \end{aligned}$$

where the sum is taken over all permutations σ of the set $\{1, \ldots, d\}$. If (e_i) is a basis for V then we conclude that f is completely determined by its value on $(e_1, \ldots, e_d)$. This shows that $\dim A_d(V) \leq 1$.

To construct a non-trivial element of V, let (e_i) be a basis for V and (α_i) the dual basis for V^*, so that $\alpha_j(e_i) = \delta_{ij}$. Define $f \in A_d(V)$ by

$$f(v_1, \ldots, v_d) := \det(\alpha_i(v_j)).$$

Then $f(e_1, \ldots, e_d) = 1$, hence $\dim A_d(V) \geq 1$. $\square$

The **wedge product**

$$A_k(V) \times A_\ell(V) \to A_{k+\ell}, \quad (f, g) \mapsto f \wedge g$$

is a bilinear map defined for all non-negative integers k, ℓ, see [14, 9]. We define it here for $k = \ell = 1$. Given $f, g \in A_1(V) = V^*$ we define $f \wedge g \in A_2(V)$ by

$$(f \wedge g)(v, w) := f(v)g(w) - f(w)g(v). \tag{7.1}$$

If $T : V \to W$ is a linear map between finite-dimensional vector spaces, and $f \in A_k(W)$, then we define the **pull-back** $T^* f \in A_k(V)$ by

$$(T^* f)(v_1, \ldots, v_k) := f(Tv_1, \ldots, Tv_k).$$

It is easy to see that

$$T^*(f \wedge g) = (T^* f) \wedge (T^* g)$$

for all $f, g \in A_1(W)$, and in fact the same holds for alternating tensors f, g on W of any degrees.

7.2 Differential forms

For $k \geq 0$, a **differential form** on M of degree k is a rule ϕ that assigns to every point $p \in M$ an alternating k–tensor ϕ_p on the tangent space T_pM. Differential forms of degree k are often called k–**forms**. Note that a 0–form

is just a real-valued function. A k–form ϕ on M is **smooth** if for all smooth vector fields $X_1, \ldots, X_k$ on M the function

$$\phi(X_1, \ldots, X_k) : M \to \mathbb{R}, \quad p \mapsto \phi_p((X_1)_p, \ldots, (X_k)_p)$$

is smooth. The set $\Omega^k(M)$ of all smooth k–forms on M is a module over the ring $C^\infty(M)$ of smooth functions on M.

For $k, \ell \geq 0$ the **wedge product**

$$\Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M), \quad (\phi, \psi) \mapsto \phi \wedge \psi$$

is a $C^\infty(M)$–bilinear map defined pointwise by

$$(\phi \wedge \psi)_p := \phi_p \wedge \psi_p.$$

Note that if ϕ, ψ are 1–forms and X, Y smooth vector fields on M then

$$(\phi \wedge \psi)(X, Y) = \phi(X)\psi(Y) - \phi(Y)\psi(X).$$

If $h : M \to M'$ is a smooth map between manifolds and ϕ is a k–form on M' then the **pull-back** $h^*\phi$ is the k–form on M defined by

$$(h^*\phi)_p := (dh_p)^*\phi_{h(p)}.$$

Thus, if $v_1, \ldots, v_k \in T_pM$ then

$$(h^*\phi)_p(v_1, \ldots, v_k) = \phi_{h(p)}(dh_p(v_1), \ldots, dh_p(v_k)).$$

One can show that if ϕ is smooth then $h^*\phi$ is smooth, see [14, 9].

The **exterior derivative**

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

is a real-linear map defined for all $k \geq 0$, see [14, 9]. We define it here for $k = 0, 1$.

Given $f \in \Omega^0(M) = C^\infty(M)$, the 1–form df on M is defined by

$$(df)_p(v) := df_p(v),$$

for $p \in M$, $v \in T_pM$. Here, $df_p : T_pM \to \mathbb{R}$ is the differential of f at p. For any smooth vector field X on M we then have

$$(df)(X) = \partial_X f.$$

We will now define the exterior derivative of 1–forms.

Proposition 7.2.1 *For any smooth* 1*–form* α *on* M *there is a unique smooth* 2*–form* $d\alpha$ *on* M *such that for all smooth vector fields* X, Y *on* M *one has*

$$d\alpha(X,Y) = \partial_X(\alpha(Y)) - \partial_Y(\alpha(X)) - \alpha([X,Y]). \tag{7.2}$$

Proof. We claim that right-hand side of Equation (7.2) defines a $C^\infty(M)$–bilinear map

$$B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M).$$

Given this, we can complete the proof of the proposition by arguing as in the proof of Proposition 5.2.3.

The map B is obviously biadditive. Now let $f \in C^\infty(M)$. Then

$$\begin{aligned} B(fX,Y) &= \partial_{fX}\alpha(Y) - \partial_Y\alpha(fX) - \alpha([fX,Y]) \\ &= f\partial_X\alpha(Y) - \partial_Y(f\cdot\alpha(X)) - \alpha(f[X,Y] - \partial_Y f\cdot X) \\ &= f\partial_X\alpha(Y) - \partial_Y f\cdot\alpha(X) - f\partial_Y\alpha(X) - f\alpha([X,Y]) + \partial_Y f\cdot\alpha(X) \\ &= f\cdot B(X,Y). \end{aligned}$$

Because B is skew-symmetric, we also have $B(X,fY) = f\cdot B(X,Y)$. □

Proposition 7.2.2 *For any* $f, g \in \Omega^0(M)$ *one has*

$$d(fdg) = df \wedge dg.$$

Proof. For all smooth vector fields X, Y on M one has

$$\begin{aligned} [d(fdg)](X,Y) &= \partial_X(f\partial_Y g) - \partial_Y(f\partial_X g) - f\partial_{[X,Y]}g \\ &= \partial_X f\cdot\partial_Y g + f\partial_X\partial_Y g - \partial_Y f\cdot\partial_X g - f\partial_Y\partial_X g - f\partial_{[X,Y]}g \\ &= (df\wedge dg)(X,Y), \end{aligned}$$

where in the last equation we used Proposition 4.2.2, which holds on any manifold. □

Let $x^1, \ldots, x^k$ be standard coordinates on $\mathbb{R}^k$. The ith coordinate x^i is a smooth map $\mathbb{R}^n \to \mathbb{R}$ whose differential $dx^i \in \Omega^1(\mathbb{R}^k)$ is given by

$$(dx^i)_p(v) = v^i$$

for any tangent vector $v = (v^1, \ldots, v^k) \in T_p\mathbb{R}^k = \mathbb{R}^k$. On an open subset $U \subset \mathbb{R}^k$, any smooth 1–form α therefore has the form

$$\alpha = \sum_i f_i dx^i$$

for some $f_i \in C^\infty(U)$, and by Proposition 7.2.2 we have

$$d\alpha = \sum_i df_i \wedge dx^i.$$

7.3 Volume forms

Recall that the parallelepiped P spanned by three vectors $a, b, c \in \mathbb{R}^3$ has volume

$$\mathrm{Vol}(P) = |\det(a, b, c)|.$$

More generally, let $v_1, \ldots, v_n \in \mathbb{R}^n$. Then the block

$$B := \left\{ \sum_i t_i v_i \;\middle|\; 0 \le t_i \le 1 \right\}$$

spanned by these vectors has n–dimensional volume

$$\mathrm{Vol}(B) = |\det(v_1, \ldots, v_n)|.$$

(An elementary proof of this is given in [7, Chapter VIII, §2].) For this reason, the function that assigns to the n–tuple $(v_1, \ldots, v_n)$ the determinant $\det(v_1, \ldots, v_n)$ is called the **volume tensor** on $\mathbb{R}^n$. The following lemma generalizes this notion so that it can be applied to tangent spaces of manifolds.

By a (real) **inner product space** we mean a real vector space equipped with an inner product.

Lemma 7.3.1 *If V is an n–dimensional oriented inner product space then there exists a unique alternating n–tensor f on V, called the* **volume tensor***, such that for every positively oriented orthonormal basis $(e_1, \ldots, e_n)$ for V one has*

$$f(e_1, \ldots, e_n) = 1.$$

Proof. As we saw in the proof of Lemma 7.1.1, an alternating n–tensor on V is completely determined by its value on any basis for V. This proves the uniqueness part of the lemma.

To prove existence, let $(e_1, \ldots, e_n)$ be a positively oriented orthonormal basis for V and define $f \in A_n(V)$ by

$$f(v_1, \ldots, v_n) := \det(\langle v_i, e_j \rangle).$$

If $(e'_1, \ldots, e'_n)$ is another positively oriented orthonormal basis for V then $e'_j = \sum_i a_{ij} e_i$ for some orthogonal matrix $A = (a_{ij})$ of positive determinant. Since every orthogonal matrix has determinant ± 1, we must have $\det(A) = 1$. Therefore,

$$f(e'_1, \ldots, e'_n) = \det(A) \cdot f(e_1, \ldots, e_n) = 1. \qquad \square$$

If M is an oriented n–manifold, the **volume form** on M is the n–form μ on M such that μ_p is the volume tensor on the tangent space T_pM for every $p \in M$. To see that μ is smooth, note that if $(E_1, \ldots, E_n)$ is a positively oriented orthonormal frame on an open subset $V \subset M$ and $X_1, \ldots, X_n$ are smooth vector fields on V then

$$\mu(X_1, \ldots, X_n) = \det(\langle X_i, E_j \rangle),$$

which is a smooth function on V.

Example 7.3.1 The volume form on $\mathbb{R}^n$ is

$$\mu = dx^1 \wedge \cdots \wedge dx^n,$$

where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$. For $n = 1$ this is trivial, and for $n = 2$ it follows from the description (7.1) of the wedge product of 1–forms. For $n \geq 3$, see [14, 9].

To prepare for the integration theory in Section 7.5, we would like to understand how a local parametrization $F : U \to M$ distorts volume locally. If M is oriented with volume form μ then the distortion is measured by the pull-back $F^*\mu$. In order to compute the latter, we will use the following lemma.

Lemma 7.3.2 *Let V, V' be oriented n–dimensional inner product spaces with volume tensors f, f' respectively. Let $T : V \to V'$ be an isomorphism of vector spaces and suppose T preserves orientation. Let $(e_1, \ldots, e_n)$ be an orthonormal basis for V and define the matrix $G = (g_{ij})$ by*

$$g_{ij} := \langle Te_i, Te_j \rangle.$$

Then

$$T^* f' = \sqrt{\det(G)} \cdot f.$$

Proof. Let $(e'_1, \ldots, e'_n)$ be an orthonormal basis for V' which is positively (or negatively) oriented according as $(e_1, \ldots, e_n)$ is positively (or negatively) oriented. Let $A = (a_{ij})$ be the matrix of T, so that

$$Te_j = \sum_i a_{ij} e'_i.$$

Then

$$g_{ij} = \langle \textstyle\sum_k a_{ki} e'_k, \sum_\ell a_{\ell j} e'_\ell \rangle = \sum_k a_{ki} a_{kj},$$

hence $G = A^T A$. This yields

$$\det(G) = \det(A^T)\det(A) = (\det(A))^2.$$

By assumption, $\det(A) > 0$, so $\det(A) = \sqrt{\det(G)}$. Now,

$$\begin{aligned}(T^* f')(e_1, \ldots, e_n) &= f'(Te_1, \ldots, Te_n) \\ &= \det(A) \cdot f'(e_1', \ldots, e_n') \\ &= \sqrt{\det(G)} \cdot f(e_1, \ldots, e_n),\end{aligned}$$

hence $T^* f' = \sqrt{\det(G)} \cdot f$. □

Proposition 7.3.1 *Let M be an oriented n–manifold with volume form μ. Let $F : U \to M$ be an orientation preserving local parametrization. Then*

$$F^*\mu = \sqrt{\det(\tilde{G})}\, dx^1 \wedge \cdots \wedge dx^n,$$

where $\tilde{G} = (\tilde{g}_{ij})$ is the matrix of the fundamental form with respect to F.

Proof. Apply Lemma 7.3.2 to the differential $dF_u : \mathbb{R}^n \to T_{F(u)}M$. □

Proposition 7.3.2 *For $i = 1, 2$ let $F_i : U_i \to M$ be a local parametrization and $\tilde{G}_i$ the corresponding matrix of the first fundamental form. Let $V_i := F_i(U_i)$ and*

$$\phi := F_2^{-1} \circ F_1 : F_1^{-1}(V) \to F_2^{-1}(V).$$

The for every $x \in F_1^{-1}(V)$ one has

$$\det(\tilde{G}_1(x)) = det(\tilde{G}_2(\phi(x))) \cdot |\det(d\phi_x)|^2.$$

Proof. This is easily deduced from the fact that, in the proof of Lemma 7.3.2, the equation $\det(G) = \det(A)^2$ holds without the assumption that T preserves orientations. □

7.4 Connection forms

Let M be an n–manifold. To any smooth frame $(E_1, \ldots, E_n)$ on an open subset $V \subset M$ we can associate an $n \times n$ matrix (ω_j^i) of smooth 1–forms on V called **connection forms**. These are uniquely determined by the fact that

$$\nabla_X E_j = \sum_i \omega_j^i(X) \cdot E_i$$

for any vector field X on V.

Lemma 7.4.1 *If the frame $(E_1, \ldots, E_n)$ is orthonormal then the matrix (ω^i_j) is skew-symmetric, i.e.*

$$\omega^j_i = -\omega^i_j$$

for all i, j.

Proof. Because $\langle E_i, E_j \rangle$ is a constant function on V we have

$$0 = \partial_X \langle E_i, E_j \rangle = \langle \nabla_X E_i, E_j \rangle + \langle E_i, \nabla_X E_j \rangle = \omega^j_i(X) + \omega^i_j(X). \qquad \square$$

Now suppose M has dimension 2. In this case the matrix (ω^i_j) is completely determined by the element ω^1_2, which we simply denote by ω and refer to as the **connection form** of the frame. For any vector field X on V we then have

$$\begin{aligned} \nabla_X E_1 &= \omega^2_1(X) E_2 = -\omega(X) E_2, \\ \nabla_X E_2 &= \omega^1_2(X) E_1 = \omega(X) E_1, \end{aligned} \tag{7.3}$$

which yields

$$\omega(X) = \langle \nabla_X E_2, E_1 \rangle.$$

Proposition 7.4.1 *Let S be an oriented regular surface with Gauss curvature K and volume form μ. Let (E_1, E_2) be a positively oriented, orthonormal frame on an open subset $V \subset S$ and ω the corresponding connection form. Then*

$$d\omega = K\mu.$$

Proof. For any smooth vector fields X, Y on V we have

$$\begin{aligned} d\omega(X, Y) &= \partial_X \omega(Y) - \partial_Y \omega(X) - \omega([X, Y]) \\ &= \partial_X \langle \nabla_Y E_2, E_1 \rangle - \partial_Y \langle \nabla_X E_2, E_1 \rangle - \langle \nabla_{[X,Y]} E_2, E_1 \rangle \\ &= \langle \nabla_X \nabla_Y E_2, E_1 \rangle + \langle \nabla_Y E_2, \nabla_X E_1 \rangle \\ &\quad - \langle \nabla_Y \nabla_X E_2, E_1 \rangle - \langle \nabla_X E_2, \nabla_Y E_1 \rangle - \langle \nabla_{[X,Y]} E_2, E_1 \rangle \\ &= \langle R(X, Y) E_2, E_1 \rangle. \end{aligned}$$

By Theorem 5.3.2 we therefore have

$$K = \langle R(E_1, E_2) E_2, E_1 \rangle = d\omega(E_1, E_2).$$

By Lemma 7.1.1 we can write $d\omega = f\mu$ for some real-valued function f on V. Then $f = d\omega(E_1, E_2) = K$, and the proposition is proved. $\square$

7.5 Integration

We will consider two kinds of integrals over an n–manifold M. The first kind is integrals $\int_M \phi$ of n–forms, which requires an orientation of M but makes no reference to the first fundamental form. The second kind is integrals $\int_M f dV$ of real-valued functions, which does not require an orientation but depends on the first fundamental form.

For integration on $\mathbb{R}^n$ we will use the Lebesgue integral, although the Riemann integral would suffice for our applications in Chapter 8. If $U \subset \mathbb{R}^n$ is an open subset, a (Lebesgue) measurable function $h : U \to \mathbb{R}$ is called **integrable** if

$$\int_U |h(x)|\, dx < \infty,$$

where dx is Lebesgue measure on U.

The integration theory on manifolds rests on the change of variables formula, a version of which we now state.

Theorem 7.5.1 *Let $F : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. If $h : V \to \mathbb{R}$ an integrable function then $(h \circ F) \cdot |\det(dF)|$ is an integrable function on U, and*

$$\int_U h(F(x)) \cdot |\det(dF_x)|\, dx = \int_V h(x)\, dx.$$

A proof can be found in [2, p. 74].

We will now consider which functions can be integrated over a manifold M. For simplicity, we will assume that M can be covered by finitely many coordinate domains. (This will be the case, for instance, if M is compact or if M is diffeomorphic to the interior of a compact manifold-with-boundary.) We then call a function $h : M \to \mathbb{R}$ **integrable** if, for every local parametrization $F : U \to M$ with corresponding matrix $\tilde{G}$, then function $(h \circ f) \cdot \sqrt{\tilde{G}}$ on U is integrable. Let $L^1(M)$ denote the real vector space consisting of all integrable functions on M.

Theorem 7.5.2 *Suppose the manifold M can be covered by finitely many coordinate domains. Then there is a unique real linear map*

$$L^1(M) \to \mathbb{R}, \quad h \mapsto \int_M h\, dV$$

such that if $F : U \to M$ is a local parametrization with corresponding matrix $\tilde{G}$, and h vanishes outside $F(U)$, then

$$\int_M h\, dV = \int_U h(F(x)) \sqrt{\tilde{G}(x)}\, dx. \tag{7.4}$$

Before giving the proof, we make one definition. For any subset $A \subset M$ let χ_A be the **characteristic function** of A, i.e.

$$\chi_A(p) = \begin{cases} 1, & p \in A, \\ 0, & p \in M - A. \end{cases}$$

Proof of Theorem 7.5.2: We first prove existence. If h vanishes outside the image of a local parametrization $F : U \to M$ we define the integral of h by (7.4). It follows from Proposition 7.3.2 and the change of variables formula, Theorem 7.5.1, that $\int_M h\, dV$ does not depend on the choice of F.

To define the integral of an arbitrary element of $L^1(M)$, we invoke the assumption that there exist finitely many local parametrizations $F_i : U_i \to M$, $i = 1, \ldots, r$, such that if $V_i = F(U_i)$ then

$$M = \bigcup_i V_i.$$

We define subsets $A_1, \ldots, A_r$ of M recursively by

$$\begin{aligned} A_1 &:= V_1, \\ A_k &:= V_k - \bigcup_{i=1}^{k-1} V_i, \quad 1 < k \leq r. \end{aligned}$$

Then $A_i \subset V_i$, $M = \bigcup_i A_i$, and $A_i \cap A_j = \emptyset$ if $i \neq j$. For an arbitrary $h \in L^1(M)$ we now define

$$\int_M h\, dV := \sum_i \int_M h \cdot \chi_{A_i} dV,$$

where the right-hand side is defined as above. To show that the integral of h is independent of the choice of the sets A_i, suppose $B_1, \ldots, B_s$ is another collection of mutually disjoint Borel measurable subsets of M such that

$$B_i \subset V_i, \quad \bigcup_i B_i = M.$$

Because

$$\sum_i \chi_{A_i} = 1 = \sum_j \chi_{B_j},$$

we have

$$\sum_i \int_M h \cdot \chi_{A_i} dV = \sum_{ij} \int_M h \cdot \chi_{A_i} \cdot \chi_{B_j} dV = \sum_j \int_M h \cdot \chi_{B_j} dV,$$

proving that the integral of h is well-defined.

The uniqueness part of the theorem follows from the linearity of the integral and the fact that $h = \sum_i h \cdot \chi_{A_i}$ for any $h \in L^1(M)$. $\square$

If M has dimension 1 or 2 then we usually write $\int_M h\, ds$ or $\int_M h\, dA$, respectively, instead of $\int_M h\, dV$.

Now suppose M is an oriented n–manifold with volume form μ. We again assume that M can be covered by finitely many coordinate domains. By Lemma 7.1.1, any n–form ω on M can be written as $\omega = h\mu$ for some real-valued function h on M. We say ω is **integrable** if h is integrable, and in that case we define

$$\int_M \omega := \int_M h\, dV. \tag{7.5}$$

The following proposition shows that the integral of n–forms on M does not depend on the first fundamental form.

Proposition 7.5.1 *Let M be an oriented n–manifold that can be covered by finitely many coordinate domains, and let ω be an integrable n–form on M that vanishes outside the image of an orientation preserving local parametriation $F : U \to M$. Then*

$$F^*\omega = f \cdot dx^1 \wedge \cdots \wedge dx^n$$

for some integrable function f on U, and

$$\int_M \omega = \int_U f(x)\, dx.$$

Proof. This follows from the expression for $F^*\mu$ given in Proposition 7.3.1. $\square$

We now consider integrals over a compact subset $R \subset M$. The set R is contained in the union of finitely many coordinate domains $V_1, \ldots, V_k$. Let $V = \cup_i V_i$. For real-valued functions f on M we define

$$\int_R f\, dV := \int_V \chi_R f,$$

where the right-hand side is as in Theorem 7.5.2. Similarly, if M is oriented then for n–forms ω on M we define

$$\int_R \omega := \int_V \chi_R \omega,$$

where the right-hand side is as in (7.5).

7.6 Stokes' theorem for surfaces

Let S be a regular surface. By a **smooth region** in S we mean a compact subset $R \subset S$ such that the following hold.

- R is the closure (in S) of an open subset of S.
- The boundary ∂R of R (as a subset of S) is a 1–manifold.

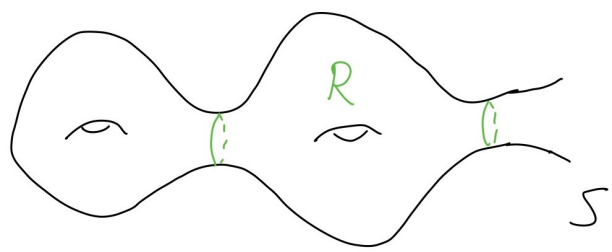


Figure 7.1: A smooth region

Now suppose S is oriented. The boundary ∂R then inherits an orientation from (S, R) called the **boundary orientation**. To define this, let $p \in S$ and choose a non-zero tangent vector $w \in T_pS$ which is inward-pointing with respect to R. (One can show that R is a 2–manifold-with-boundary, and a precise definition of inward-pointing is then given in [14].) A non-zero tangent vector $v \in T_p(\partial R)$ is **positively oriented** if (v, w) is a positively oriented basis for T_pS. This definition is independent of the choice of w, see [14].

Let $\iota : \partial R \to S$ be the inclusion map. For any $\omega \in \Omega^1(S)$ we define

$$\int_{\partial R} \omega := \int_{\partial R} \iota^* \omega,$$

where the right-hand side is as defined in Section 7.5.

Theorem 7.6.1 (Stokes) *Let S be an oriented regular surface and $R \subset S$ a smooth region. For any $\omega \in \Omega^1(S)$ one then has*

$$\int_{\partial R} \omega = \int_R d\omega.$$

Proofs can be found in [14, 9].

If $S = \mathbb{R}^2$ with coordinates x, y then $\omega = f\,dx + g\,dy$ for some smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, and

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

Stokes' theorem now says that

$$\int_{\partial R} (f\,dx + g\,dy) = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy,$$

which is an instance of Green's theorem.

In the situation of Theorem 7.6.1, let $C_1, \ldots, C_r$ be the components of ∂R. Each C_i is a compact, connected 1–manifold and therefore diffeomorphic to S^1. Choose a diffeomorphism $\gamma_i : S^1 \to C_i$ and let

$$\alpha_i : \mathbb{R} \to C_i, \quad t \mapsto \gamma_i(\cos t, \sin t).$$

Then

$$\int_{\partial R} \omega = \sum_{i=1}^{r} \int_0^{2\pi} \omega_{\alpha_i(t)}(\dot{\alpha}_i(t))\,dt.$$

Problems

7.1 Let M be an n–manifold and $F : U \to M$ a local parametrization. Let $(X_1, \ldots, X_n)$ be the corresponding coordinate frame on $V := F(U)$. The compononents $x^1, \ldots, x^n$ of the inverse map $F^{-1} : V \to U$ are called **local coordinates** on V. Each x^i is a real-valued function on V, and

$$F^{-1}(p) = (x^1(p), \ldots, x^n(p))$$

for $p \in V$. Show that

$$dx^i(X_j) = \delta^i_j,$$

where the right-hand side is the Kronecker delta, i.e. $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$.

7.2 This problem will introduce the covariant derivative on 1–forms on a manifold M. Recall that $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M, and $\Omega^1(M)$ denotes the space of smooth 1–forms. For $X \in \mathfrak{X}(M)$

and $\alpha \in \Omega^1(M)$ we will write $\langle \alpha, X \rangle$ instead of $\alpha(X)$. Show that there is a unique real linear map

$$\mathfrak{X}(M) \times \Omega^1(M) \to \Omega^1(M), \quad (X, \alpha) \mapsto \nabla_X \alpha$$

such that for any $X, Y \in \mathfrak{X}(M)$, $\alpha \in \Omega^1(M)$, and $f \in C^\infty(M)$ one has

$$\begin{aligned} \nabla_{fX} \alpha &= f \nabla_X \alpha, \\ \nabla_X (f\alpha) &= \partial_X f \cdot \alpha + f \nabla_X \alpha, \\ \partial_X \langle Y, \alpha \rangle &= \langle \nabla_X Y, \alpha \rangle + \langle Y, \nabla_X \alpha \rangle. \end{aligned}$$

Hint: For fixed X, α, show that the map

$$\mathfrak{X}(M) \to C^\infty(M), \quad Y \mapsto \partial_X \langle Y, \alpha \rangle - \langle \nabla_X Y, \alpha \rangle$$

is linear over $C^\infty(M)$.

7.3 Let M be a manifold and let $x^1, \ldots, x^n$ be local coordinates on an open subset $V \subset M$, see Problem 7.1. Let $X_1, \ldots, X_n$ be the corresponding coordinate vector fields on V. Show that for any smooth 1–form α on V one has

$$d\alpha = \sum_{i=1}^n dx^i \wedge \nabla_{X_i} \alpha,$$

where we use the covariant derivative on 1–forms introduced in Problem 7.2.

Hint: Evaluate both sides on (X_j, X_k).

7.4 Let S be an oriented regular surface. For any $p \in S$ let

$$J : T_pS \to T_pS$$

be the rotation with angle $\pi/2$ in the positive direction. In other words, if $0 \neq v \in T_pS$ then (v, Jv) is a positively oriented basis for T_pS. We can now make T_pS a complex vector space by defining

$$(a + bi) \cdot v := av + bJv$$

for any real numbers a, b and $v \in T_pS$. Here, $i^2 = -1$. Prove that for any smooth vector fields X, Y on S and smooth function $f : S \to \mathbb{C}$ one has

$$\nabla_X (fY) = \partial_X f \cdot Y + f \nabla_X Y.$$

Hint: If E is any smooth unit vector field on an open subset $V \subset S$ then (E, JE) is a positively oriented orthonormal frame on V. Now use the

equations (7.3) and observe that any vector field on V has the form hE for some $h : V \to \mathbb{C}$.

7.5 Let S be an oriented regular surface with Gauss curvature K. Let $R \subset S$ be a smooth region with connected, non-empty boundary C. Given $p \in C$, show that parallel transport once around C from p to p in the positive direction is the rotation with angle η, where

$$\eta = \int_R K \, dA.$$

Hint: One can show that there exists a smooth unit vector field E on some open subset $V \subset S$ containing R. (You may assume this.)

Chapter 8

The Gauss-Bonnet theorem

8.1 Winding numbers

In this section we will state the Hopf *Umlaufsatz*, or rotation index theorem, which will be used in the proof of the Gauss Bonnet theorem.

A continuous, non-constant curve $c : \mathbb{R} \to \mathbb{R}^n$ is called **periodic** if there exists a positive real number λ such that

$$c(t + \lambda) = c(t)$$

for all t. The smallest such λ is then called the **period** of c.

Example 8.1.1 The plane curve $c(t) = (\cos t, \sin t)$ has period 2π.

For given $L > 0$, curves $c : \mathbb{R} \to \mathbb{R}^n$ of period L are in one-to-one correspondence with maps $f : S^1 \to \mathbb{R}^n$ through the relation

$$c(t) = f(e^{2\pi i t/L}).$$

Moreover, c is smooth if and only if f is smooth. If f is injective, or equivalently if c restricts to an injective map $[0, L) \to \mathbb{R}^n$, then c is called **simple periodic**. In this case, f is a topological embedding. If in addition c is regular then one can show that f is a diffeomorphism onto a 1–manifold, see [14, 9].

We will make use of the complex exponential function e^z. Recall that if $z = x + iy$ for real numbers x, y then

$$e^z = e^x(\cos y + i \sin y).$$

Lemma 8.1.1 *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{C} - \{0\}$ a continuously differentiable function.*

(i) There exists a continuously differentiable function $g : I \to \mathbb{C}$ such that $f(t) = e^{g(t)}$ for all $t \in I$.

(ii) If g_1, g_2 are two functions as in (i) then

$$g_1 - g_2 = 2\pi i k$$

for some constant $k \in \mathbb{Z}$.

Proof. Choose $t_0 \in I$ and a complex number a such that $f(t_0) = e^a$. To prove (ii), suppose $f = e^g$. Then

$$g(t_0) = a + 2\pi i k$$

for some integer k. Moreover,

$$\dot{f} = \dot{g} e^g = \dot{g} f,$$

so $\dot{g} = \dot{f}/f$. Therefore,

$$g(t) = g(t_0) + \int_{t_0}^{t} \dot{g} = a + 2\pi i k + \int_{t_0}^{t} \frac{\dot{f}}{f},$$

proving (ii).

To prove (i), define

$$g(t) := a + \int_{t_0}^{t} \frac{\dot{f}}{f}.$$

Then $\dot{g} = \dot{f}/f$. Writing $h := f e^{-g}$ we have

$$\dot{h} = \dot{f} e^{-g} - f \dot{g} e^{-g} = 0,$$

hence h is constant. Because $h(t_0) = 1$, we have $h \equiv 1$, so $f = e^g$. $\square$

Let $c : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable curve with period L, and z_0 a complex number not in the image of c. The **winding number** $W(c; z_0)$ of c with respect to z_0 is defined as follows. By Lemma 8.1.1 we can find a continuously differentiable curve $g : \mathbb{R} \to \mathbb{C}$ such that

$$c(t) = z_0 + e^{g(t)}$$

for all t. Then

$$g(t + L) = g(t) + 2\pi i k$$

for some constant integer k, and we define $W(c; z_0) := k$. Part (ii) of the lemma shows that this definition is independent of the choice of g.

Note that if $c(t) = z_0 + r(t)e^{i\theta(t)}$ for real-valued functions r, θ with $r > 0$ then

$$W(c; z_0) = \frac{1}{2\pi}(\theta(L) - \theta(0)).$$

Let $c : \mathbb{R} \to \mathbb{C}$ be a regular, periodic curve. The **rotation index** n_c of c (also called the **tangent winding number**) is the winding number of the derivative $\dot{c} : \mathbb{R} \to \mathbb{C}$ with respect to the origin, i.e.

$$n_c := W(\dot{c}; 0).$$

If c is in fact simple periodic then one can show that its image C is a submanifold of $\mathbb{R}^2$ diffeomorphic to S^1. The Jordan curve theorem then asserts that the complement $\mathbb{R}^2 - C$ has exactly two connected components, and C is their common boundary. (A proof of the more general Jordan-Brouwer separation theorem can be found in [4, p. 89].) Moreover, one component (the "inside") is bounded, whereas the other one (the "outside") is unbounded. We say c is **positively oriented** if it is positively oriented with respect to the closure R of the bounded component.

Theorem 8.1.1 (Hopf) *Any positively oriented, regular, simple periodic curve in the plane has rotation index* 1.

For the proof we refer to [1, 10].

8.2 Geodesic curvature

Let S be a regular surface and $\gamma : I \to S$ a regular curve. By a **normal orientation** of γ we mean a smooth map $N : I \to S^2$ such that $N(t) \in T_{\gamma(t)}S$ and $N(t) \perp \dot{\gamma}(t)$ for all t. (In particular, N is a vector field on S along γ.) Such a normal orientation always exists, see Problem 8.3.

Now let $\gamma : I \to S$ be a smooth curve of unit speed and with normal orientation N. Because

$$0 = \frac{d}{dt}\|\dot{\gamma}(t)\|^2 = 2\left\langle \frac{\nabla}{dt}\dot{\gamma}(t), \dot{\gamma}(t) \right\rangle,$$

there is a unique smooth function $\kappa_\gamma : I \to \mathbb{R}$, the **geodesic curvature** of γ, such that

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \kappa_\gamma(t) \cdot N(t)$$

for all t. Clearly, $\kappa_\gamma \equiv 0$ if and only if γ is a geodesic.

The following lemma says that the geodesic curvature is invariant under reparametrization in a certain sense.

Lemma 8.2.1 *Let $I, J \subset \mathbb{R}$ be intervals. Let $\gamma_1 : I \to S$ be a smooth curve of unit speed and with normal orientation N. Suppose $\gamma_2 = \gamma_1 \circ \phi : J \to S$ is a reparametrization of γ_1 of unit speed, where $\phi : J \to I$ is smooth. Let γ_2 have the normal orientation $N_2(t) := N_1(\phi(t))$. Then the geodesic curvatures of γ_1, γ_2 are related by*

$$\kappa_{\gamma_2}(t) = \kappa_{\gamma_1}(\phi(t)).$$

Proof. It is easy to see that $\phi(t) = \epsilon t + a$ for some constants $\epsilon = \pm 1$, $a \in \mathbb{R}$, so that

$$\gamma_2(t) = \gamma_1(\epsilon t + a).$$

Hence,

$$\dot{\gamma}_2(t) = \epsilon \dot{\gamma}_1(\epsilon t + a), \quad \ddot{\gamma}_2(t) = \ddot{\gamma}_1(\epsilon t + a).$$

This yields

$$\kappa_{\gamma_2}(t) N_2(t) = \frac{\nabla}{dt} \dot{\gamma}_2(t) = \left. \frac{\nabla}{ds} \right|_{s=\phi(t)} \dot{\gamma}_1(s) = \kappa_{\gamma_1}(\phi(t)) N_1(\phi(t)),$$

from which the lemma follows. □

Corollary 8.2.1 *Let S be a regular surface and $R \subset S$ a smooth domain. There is a unique smooth function $\kappa_g : \partial R \to \mathbb{R}$ with the following property. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve of unit speed such that $\gamma(t) \in \partial R$ for every t. If γ is given the inward-pointing normal orientation with respect to R then*

$$\kappa_g(\gamma(0)) = \kappa_\gamma(0).$$

Proof. Let γ_1, γ_2 be smooth curves of unit speed taking values on ∂R, both defined in open intervals containing 0. Then $\phi := \gamma_2^{-1} \circ \gamma_1$ is defined and smooth on a neighbourhood of 0. Now apply the lemma. □

8.3 The local Gauss-Bonnet theorem, I

A smooth region R is called **simple** if it is contained in a coordinate domain for S and ∂R has exactly one component.

Theorem 8.3.1 *Let S be a regular surface with Gauss curvature K. Suppose $R \subset S$ is a simple smooth region. Then*

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds = 2\pi.$$

Proof. Let $F : U \to S$ be a local parametrization with $R \subset F(U)$. Let X_1, X_2 be the corresponding coordinate vector fields and (E_1, E_2) the orthonormal frame on $F(U)$ obtained from (X_1, X_2) by the Gram-Schmidt process. We give $F(U)$ the orientation for which F is orientation preserving. Combining Proposition 7.4.1 and Stokes's theorem we find that

$$\int_R K\, dA = \int_R d\omega = \int_{\partial R} \omega, \tag{8.1}$$

where $\omega \in \Omega^1(F(U))$ is the connection form of the frame (E_1, E_2). To compute the line integral, choose a smooth, simply periodic curve $\gamma : \mathbb{R} \to S$ of unit speed whose trace equals ∂R. Let $N : \mathbb{R} \to S^2$ be the inward-pointing normal orientation of γ. By replacing $\gamma(t)$ by $\gamma(-t)$ if necessary, we can arrange that γ is positively oriented.

We can write

$$\dot\gamma(t) = \sum_i \beta^i(t) E_i(\gamma(t))$$

where each β^i is a smooth funtion $\mathbb{R} \to \mathbb{R}$. Then $\beta := (\beta^1, \beta^2)$ is a smooth curve in the plane taking values on the unit circle. By Lemma 8.1.1 there is a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta(t) = (\cos\theta(t), \sin\theta(t))$$

for all t. Then

$$\dot\gamma(t) = \cos\theta(t) E_1(\gamma(t)) + \sin\theta(t) E_2(\gamma(t)).$$

Furthermore the normal orientation of γ is

$$N(t) = -\sin\theta(t) E_1(\gamma(t)) + \cos\theta(t) E_2(\gamma(t)),$$

as one can verify by computing $\mu_{\gamma(t)}(\dot\gamma(t), N(t)) = 1$, where μ is the volume form on S. Now,

$$\begin{aligned}\frac{\nabla}{dt}\dot\gamma(t) = &-\dot\theta(t)\sin\theta(t) E_1(\gamma(t)) + \cos\theta(t)\nabla_{\gamma(t),\dot\gamma(t)} E_1 \\ &+ \dot\theta(t)\cos\theta(t) E_2(\gamma(t)) + \sin\theta(t)\nabla_{\gamma(t),\dot\gamma(t)} E_2.\end{aligned}$$

Inserting

$$\nabla_{\gamma(t),\dot\gamma(t)} E_1 = -\omega_{\gamma(t)}(\dot\gamma(t)) E_2(\gamma(t)),$$
$$\nabla_{\gamma(t),\dot\gamma(t)} E_2 = \omega_{\gamma(t)}(\dot\gamma(t)) E_1(\gamma(t)),$$

we get

$$\frac{\nabla}{dt}\dot\gamma(t) = \Big(\dot\theta(t) - \omega_{\gamma(t)}(\dot\gamma(t))\Big) N(t).$$

Hence, the geodesic curvature of γ is

$$\kappa_\gamma(t) = \dot\theta(t) - \omega_{\gamma(t)}(\dot\gamma(t)). \tag{8.2}$$

If γ has period L then this yields

$$\int_{\partial R} \omega = \int_0^L \Big(\dot\theta(t) - \kappa_\gamma(t)\Big)\, dt = \theta(L) - \theta(0) - \int_{\partial R} \kappa_g ds.$$

Combining this with (8.1) we obtain

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds = 2\pi W(\beta; 0).$$

It only remains to prove that the winding number $W(\beta; 0) = 1$. To this end we compare γ with the plane curve $\alpha := F^{-1} \circ \gamma$. Clearly, α is regular and simple periodic. Let $\alpha = (\alpha^1, \alpha^2)$. Since $\gamma = F \circ \alpha$, the chain rule yields

$$\dot\gamma(t) = \sum_i \dot\alpha^i(t) \partial_i F(\alpha(t)) = \sum_i \dot\alpha^i(t) X_i(\gamma(t)). \tag{8.3}$$

For fixed t, and omitting t and γ from notation for a moment, we then have

$$\dot\gamma = \sum_i \dot\alpha^i X_i = \sum_i \beta^i E_i.$$

Recall that the Gram-Schmidt process transforms a basis by an upper triangular matrix with positive entries on the diagonal. In our case,

$$E_j = \sum_i c^i_j X_i,$$

where the matrix $C := (c^i_j)$ satisfies $c^i_i > 0$, and $c^i_j = 0$ for $i > j$. Therefore,

$$\dot\gamma = \sum_j \beta^j \sum_i c^i_j X_i = \sum_i \left(\sum_j c^i_j \beta^j \right) X_i.$$

In view of (8.3), this shows that

$$\dot{\alpha}^i = \sum_j c^i_j \beta^j,$$

i.e.

$$\dot{\alpha} = C\beta.$$

Since the matrix C has only positive eigenvalues (namely c^i_i), we see that $\dot{\alpha}(t)$ is never a negative real multiple of $\beta(t)$. It is then a simple exercise to show that the curves $\dot{\alpha}$ and β have the same winding number with respect to the origin (see Problem 8.1). Thus,

$$W(\beta; 0) = W(\dot{\alpha}; 0) = n_\alpha = 1,$$

where the last equality is the theorem of Hopf. This completes the proof of the theorem. $\square$

As an example, let S^2 be the unit sphere in $\mathbb{R}^3$, and let $R \subset S^2$ be the upper hemisphere. Since ∂R is a great circle, which can be parametrized by a geodesic, we have $\kappa_g = 0$. Moreover, $K = 1$, so the theorem says that

$$2\pi = \int_R K\, dA + \int_{\partial R} \kappa_g\, ds = \int_R 1\, dA = \text{Area}(R) = \frac{1}{2}\text{Area}(S^2),$$

confirming that the area of S^2 is 4π.

8.4 The local Gauss-Bonnet theorem, II

Given a manifold M, a continuous curve $\gamma : I \to M$ is called **piecewise regular** if for all $a, b \in I$ with $a < b$ there exists a non-negative integer r and a partition

$$a = a_0 < a_1 < \cdots < a_r = b$$

such that the restriction of γ to the subinterval $[a_{i-1}, a_i]$ is a regular curve for $i = 1, \ldots, r$.

Let S be a regular surface. By a **polygonal region** in S we mean a compact subset $R \subset S$ such that the following hold.

- R is the closure of an open subset of S.
- The boundary ∂R has finitely many components.
- Each component of ∂R is the image of a simple periodic, piecewise regular curve $\mathbb{R} \to S$.

A polygonal region R is called **simple** if it is contained in a coordinate domain for S and ∂R has exactly one component.

Let $R \subset S$ be a polygonal region and $\gamma : \mathbb{R} \to S$ a simple periodic, piecewise regular curve of unit speed whose trace is a boundary component of R. If $t_0 \in \mathbb{R}$ is a point where γ is not smooth then $\gamma(t_0)$ is called a **vertex** of R. A vertex $\gamma(t_0)$ is called a **cusp** if the one-sided derivatives $\dot{\gamma}(t_0^{\pm})$ of γ at t_0 satisfy

$$\dot{\gamma}(t_0^+) = -\dot{\gamma}(t_0^-);$$

otherwise $\gamma(t_0)$ is called an **ordinary vertex**. If $\theta \in [0, 2\pi]$ is the interior angle of ∂R at a vertex p then $\epsilon := \pi - \theta \in [-\pi, \pi]$ is called the **jump angle** at p. If γ is smooth on a non-empty open interval (t_0, t_1) but not smooth at t_0 or at t_1 then the image of the closed interval $[t_0, t_1]$ under γ is called an **edge** of R.

If S is oriented then the jump angle has the following alternative description: If γ is orientation preserving then $\gamma(t_0^+)$ is obtained from $\gamma(t_0^-)$ by a rotation with oriented angle ϵ, see Figure 8.1.

Let $V \subset \partial R$ be the set of vertices of R, which is finite. The geodesic curvature

$$\kappa_g : \partial R - V \to \mathbb{R}$$

can then be defined as before.

Theorem 8.4.1 *Let S be a regular surface with Gauss curvature K. Suppose $R \subset S$ is a simple polygonal region with jump angles $\epsilon_1, \ldots, \epsilon_k$ at the vertices. Then*

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds + \sum_{i=1}^{k} \epsilon_i = 2\pi.$$

Idea of proof. "Round off" the corners of ∂R to produce a smooth region $R' \subset S$ to which Theorem 8.3.1 can be applied. Use Equation (8.2) to estimate the integral $\int_{\partial R'} \kappa_g\, ds$. $\square$

A simple polygonal region $R \subset S$ is called a **curved triangle** if ∂R has exactly three vertices. If in addition each edge of ∂R can be parametrized by a geodesic then R is called a **geodesic triangle**.

Theorem 8.4.2 *Let $R \subset S$ be a geodesic triangle with interior angles θ_i, $i = 1, 2, 3$. Then*

$$\int_R K\, dA = \sum_{i=1}^{3} \theta_i - \pi.$$

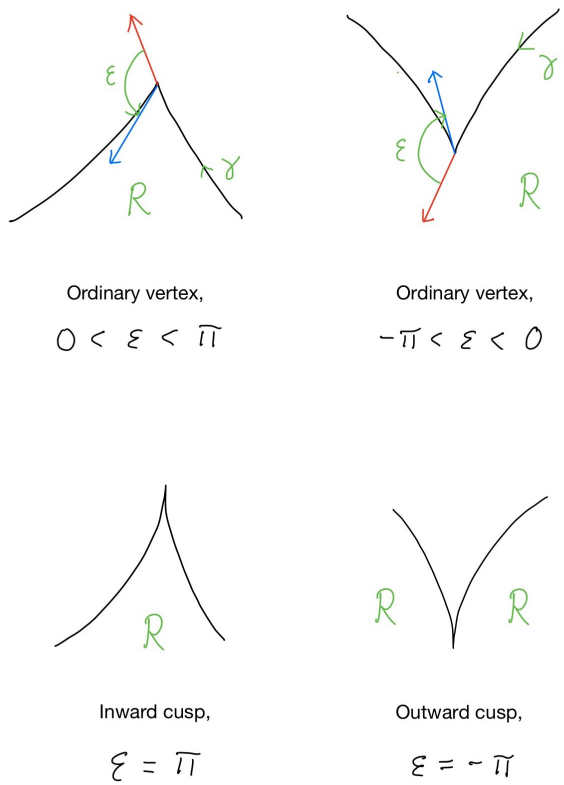


Figure 8.1: The jump angle

Proof. The jump angle at the ith vertex is $\epsilon_i = \pi - \theta_i$. Since $\kappa_g = 0$, Theorem 8.4.1 gives

$$\int_R K\,dA = 2\pi - \sum_{i=1}^{3}(\pi - \theta_i) = \sum_{i=1}^{3}\theta_i - \pi. \qquad \square$$

If K is constant then the theorem says that

$$K \cdot \mathrm{Area}(R) = \sum_{i=1}^{3}\theta_i - \pi.$$

Note that the cases $K = 0, 1, -1$ correspond to Euclidean, spherical, and hyperbolic triangles, respectively.

8.5 The global Gauss-Bonnet theorem

Let $S \subset \mathbb{R}^n$ be a regular surface (not necessarily orientable) and $R \subset S$ a polygonal region. A **smooth triangulation** of R is a finite collection $\{R_1, \ldots, R_f\}$ of curved triangles in S such that the following hold.

- The union of the triangles equals R, i.e. $\bigcup_h R_h = R$.
- If two triangles intersect non-trivally then their intersection is either a common vertex or a common edge.

Each triangle R_h is called a **face** of the triangulation, and each vertex (or edge) of a triangle is called a vertex (or edge) of the triangulation. Let v, e be the number of vertices and edges of the triangulation, respectively, and recall that there are f faces. Then

$$\chi(R) := v - e + f$$

is called the **Euler characteristic** of the triangulated region R. It is known that every polygonal region admits a smooth triangulation, and the following theorem shows that the Euler characteristic is independent of the triangulation.

Theorem 8.5.1 *Let S be a regular surface and $R \subset S$ a polygonal region. Let $\epsilon_1, \ldots, \epsilon_k$ be the jump angles at the vertices of ∂R. Then for any smooth triangulation of R one has*

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds + \sum_{i=1}^{k} \epsilon_i = 2\pi\chi(R).$$

Proof. A vertex p of the triangulation is called a **boundary vertex** if $p \in \partial R$, otherwise p is called an **interior vertex**. It is easy to see that every vertex of R must also be a vertex of the triangulation. Let $p_1, \ldots, p_k$ be the vertices of R and $p_{k+1}, \ldots, p_\ell$ the remaining boundary vertices. Let $p_{\ell+1}, \ldots, p_v$ be the interior vertices.

Let a_i be the number of edges meeting at p_i. Because every edge of a triangle has distinct end-points, we have

$$\sum_{i=1}^{v} a_i = 2e. \tag{8.4}$$

The interior angles of the faces meeting at a vertex p_i are denoted $\{\eta_{ij}\}$, where $1 \le j \le a_i - 1$ if p_i is a boundary vertex and $1 \le j \le a_i$ if p_i is an interior vertex. Let $\epsilon_{ij} := \pi - \eta_{ij}$ be the corresponding jump angle.

For $i = 1, \ldots, k$ let η_i be the interior angle of R at p_i. The jump angle at p_i is then $\epsilon_i = \pi - \eta_i$. For $i = k+1, \ldots, \ell$ we define $\eta_i := \pi$ and $\epsilon_i := 0$.

For any boundary vertex p_i we have

$$\eta_i = \sum_{j=1}^{a_i-1} \eta_{ij},$$

which gives

$$\pi - \epsilon_i = \sum_{j=1}^{a_i-1} (\pi - \epsilon_{ij}),$$

or equivalently,

$$\sum_{j=1}^{a_i-1} \epsilon_{ij} = (a_i - 2)\pi + \epsilon_i.$$

For any interior vertex p_i we have

$$2\pi = \sum_{j=1}^{a_i} \eta_{ij} = \sum_{j=1}^{a_i} (\pi - \epsilon_{ij}),$$

hence

$$\sum_{j=1}^{a_i} \epsilon_{ij} = (a_i - 2)\pi.$$

Let $\{\eta'_{hj}\}_{1 \le j \le 3}$ be the interior angles of the hth face R_h, and $\epsilon'_{hj} := \pi - \eta'_{hj}$. Applying Theorem 8.4.1 to each R_h and summing over h we get

$$\sum_{h=1}^{f} \left(\int_{R_h} K \, dA + \int_{\partial R_h} \kappa_g \, ds + \sum_{j=1}^{3} \epsilon'_{hj} \right) = 2\pi f.$$

It is easy to see that ∂R is a union of edges of the triangulation, and every edge not contained in ∂R is the intersection of two faces. Hence, in the above sum the integrals over edges not contained in ∂R come in pairs with opposite normal orientations and therefore cancel out. We then obtain

$$\int_R K \, dA + \int_{\partial R} \kappa_g \, ds + \sum_{hj} \epsilon'_{hj} = 2\pi f.$$

Because $\{\epsilon_{ij}\}$ and $\{\epsilon'_{hj}\}$ are just different ways of indexing the same set, we have

$$\begin{aligned}\sum_{hj} \epsilon'_{hj} &= \sum_{ij} \epsilon_{ij} \\ &= \sum_{i=1}^{\ell} \sum_{j=1}^{a_i-1} \epsilon_{ij} + \sum_{i=\ell+1}^{v} \sum_{j=1}^{a_i} \epsilon_{ij} \\ &= \sum_{i=1}^{v} (a_i - 2)\pi + \sum_{i=1}^{k} \epsilon_i \\ &= 2\pi(e - v) + \sum_{i=1}^{k} \epsilon_i,\end{aligned}$$

where in the last equality we used (8.4). Putting this together we get

$$\begin{aligned}\int_R K\, dA + \int_{\partial R} \kappa_g\, ds + \sum_{i=1}^{k} \epsilon_i &= 2\pi(v - e + f) \\ &= 2\pi\chi(R). \qquad \square\end{aligned}$$

Corollary 8.5.1 *If S is a compact regular surface then*

$$\int_S K\, dA = 2\pi\chi(S). \qquad \square$$

Proposition 8.5.1 *If S is a regular surface and $R \subset S$ a simple polygonal region then $\chi(R) = 1$.*

Proof. This follows from Theorems 8.4.1 and 8.5.1. $\square$

Proposition 8.5.2 *Let S be a compact regular surface and $R_1, R_2 \subset S$ smooth regions such that*

- $R_1 \cup R_2 = S$,
- $R_1 \cap R_2 = \partial R_1 = \partial R_2$.

Then

$$\chi(S) = \chi(R_1) + \chi(R_2).$$

Proof. Let $\kappa_g^i : \partial R_i \to \mathbb{R}$ be the geodesic curvature of ∂R_i as boundary of R_i. Because κ_g^1 and κ_g^2 are defined in terms of opposite normal oriantations, we have $\kappa_g^1 = -\kappa_g^2$. Applying Theorem 8.5.1 to both R_1 and R_2 we get

$$\begin{aligned} 2\pi\left(\chi(R_1) + \chi(R_2)\right) &= \int_{R_1} K\,dA + \int_{R_2} K\,dA + \int_{R_1} \kappa_g^1\,ds + \int_{R_2} \kappa_g^2\,ds \\ &= \int_S K\,dA \\ &= 2\pi\chi(S). \qquad \square \end{aligned}$$

Proposition 8.5.3 $\chi(S^2) = 2$.

Proof. One can prove this by constructing a smooth triangulation of S^2. Alternatively, let $H_+, H_- \subset S^2$ be the upper and lower closed hemispheres, respectively. Because each hemisphere is a simple smooth region, Propositions 8.5.1 and 8.5.2 yield

$$\chi(S^2) = \chi(H_+) + \chi(H_-) = 2. \qquad \square$$

We now consider the torus $T^2 := S^1 \times S^1 \subset \mathbb{R}^4$.

Proposition 8.5.4 $\chi(T^2) = 0$.

Proof. The map

$$\mathbb{R}^2 \to T^2, \quad (s,t) \mapsto (\cos s, \sin s, \cos t, \sin t)$$

is a surjective local isometry. Because the Gauss curvature of $\mathbb{R}^2$ vanishes, the same holds for T^2 (see the remarks after Theorem 5.3.3). Now apply Corollary 8.5.1. $\square$

Problems

8.1 Let $f, g : \mathbb{R} \to \mathbb{C} - \{0\}$ be smooth curves of period L such that $f(t)$ is never a negative real multiple of $g(t)$. Show that f and g have the same winding number with respect to the origin.

8.2 Let S be a regular surface and $p \in S$. Suppose α, β are smooth curves $(-\epsilon, \epsilon) \to S$ such that

- $\alpha(0) = p = \beta(0)$, and
- $\dot\alpha(0)$ and $\dot\beta(0)$ are linearly independent.

Show that there exists a local parametrization $F : U \to S$ around p such that $(0,0) \in U$ and such that for all t in a neighbourhood of 0 in $\mathbb{R}$ one has

$$\alpha(t) = F(t,0), \quad \beta(t) = F(0,t).$$

Hint: First consider the case $S = \mathbb{R}^2$. Apply the inverse function theorem to the map

$$\phi : (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \to \mathbb{R}^2, \quad (s,t) \mapsto \alpha(t) + \beta(t).$$

8.3 Let S be a regular surface and $\gamma : I \to S$ a regular curve. Prove the following.

(a) The curve γ admits a normal orientation N.

(b) If N_1 and N_2 are normal orientations of γ then either $N_1 = N_2$ or $N_1 = -N_2$.

8.4 Let S be a regular surface and $c : I \to S$ a regular curve of unit speed and with normal orientation N. Suppose W is a parallel vector field of unit length along c. Prove the following.

(a) There is a smooth function $\theta : I \to \mathbb{R}$ such that

$$W(t) = \cos\theta(t)\dot{c}(t) + \sin\theta(t)N(t).$$

(b) If θ is as in (a) then

$$\dot{\theta}(t) = -\kappa_c(t),$$

where κ_c is the geodesic curvature of c.

Chapter 9

Hyperbolic geometry

9.1 Riemannian metrics

Let M be a k–manifold. A **Riemannian metric** g on M is a rule that assigns to every $p \in M$ a (positive definite) scalar product g_p on the tangent space T_pM such that g is smooth in the following sense: If $F : U \to M$ is any local parametrization then for $i, j = 1, \ldots, k$ the function $\tilde{g}_{ij} : U \to \mathbb{R}$ given by

$$\tilde{g}_{ij}(u) := g_{F(u)}(\partial_i F(u), \partial_j F(u))$$

is smooth. The pair (M, g) is called a **Riemannian manifold** . Often one just says that M is equipped with the Riemannian metric g. A Riemannian manifold of dimension 2 is called a **Riemannian surface**. (Note that the term "Riemann surface" usually means a complex 1–manifold.)

Example 9.1.1 For any manifold $M \subset \mathbb{R}^n$, the first fundamental form $I = I_M$ is an example of a Riemannian metric.

Example 9.1.2 A Riemannian metric g on an open subset $\Omega \subset \mathbb{R}^n$ is usually specified in terms of its components $(\tilde{g}_{ij})$ with respect to the canonical parametrization $F = \mathrm{Id} : \Omega \to \Omega$. If $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$ one writes

$$g = \sum_{ij} \tilde{g}_{ij} dx^i dx^j.$$

Example 9.1.3 Let $\mathbb{U}^2 \subset \mathbb{R}^2$ be the upper half-space, i.e.

$$\mathbb{U}^2 := \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}.$$

Let g_U be the Riemannian metric on $\mathbb{U}^2$ given by

$$g_U = \frac{(dx)^2 + (dy)^2}{y^2}.$$

Then $(\mathbb{U}^2, g_U)$ is the Poincaré **upper half-space model** for the **hyperbolic plane**, see Section **??**.

If (M, g) and (M', g') are Riemannian manifolds of the same dimension, a smooth map $\phi : M \to M'$ is called a **local isometry** if for every $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}M'$ is a linear isometry, i.e. for all $v, w \in T_pM$ one has

$$g'_{\phi(p)}(d\phi_p(v), d\phi_p(w)) = g_p(v, w). \tag{9.1}$$

If both M and M' are equipped with their first fundamental forms then this definition agrees with the one given in Section 2.2. As before, a bijective local isometry is called an **isometry**. Every local isometry is a local diffeomorphism, and every isometry is a diffeomorphism. The group of all isometries of (M, g) will be denoted by $\mathrm{Isom}(M, g)$, or $\mathrm{Isom}(M)$ if g is understood.

Given a Riemannian manifold (M, g), a map $\phi : M \to \mathbb{R}^n$ is called an **isometric embedding** if the image $N := \phi(M)$ is a manifold and the map

$$(M, g) \to (N, I_N)$$

induced by ϕ is an isometry. In other words, ϕ is a smooth embedding and for every $p \in M$ and $v, w \in T_pM$ one has

$$g_p(v, w) = \langle d\phi_p(v), d\phi_p(w)\rangle.$$

Theorem 9.1.1 (J. Nash) *For any Riemannian manifold M there exists a natural number n and an isometric embedding $M \to \mathbb{R}^n$.*

The original proof can be found in [13]. A simpler proof was later obtained by M. Günther [5].

A fundamental result of Riemannian geometry is that on any Riemannian manifold (M, g) there is a unique operator

$$\tilde{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that Propositions 4.3.2 and 4.3.3 holds with $\tilde{\nabla}$ and $g(\,\cdot\,,\,\cdot\,)$ in place of ∇ and $\langle\,\cdot\,,\,\cdot\,\rangle$, respectively. The operator $\tilde{\nabla}$ is called the *Riemannian connection* or *Levi-Civita connection* . If g is the first fundamenatal form, then $\tilde{\nabla}$ agrees with the covariant derivative ∇ as we have defined it. The

Riemannian curvature tensor and the notion of a geodesic on (M, g) are defined as before, with $\tilde{\nabla}$ in place of ∇ (see [11, 10]). We will not discuss this general theory here, as our focus in this chapter will be on the hyperbolic spaces, where the Riemannian connection has a simple description analogous to that of the covariant derivative in Section 4.3.

9.2 Hyperbolic n–space

For $n \geq 1$, **Lorentzian** $(n+1)$–**space** $\mathbb{R}^{n,1}$ is the vector space $\mathbb{R}^{n+1}$ equipped with the **Lorentzian scalar product**

$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i - x^{n+1} y^{n+1}$$

for elements $x = (x^1, \ldots, x^{n+1})$, $y = (y^1, \ldots, y^{n+1})$ of $\mathbb{R}^{n,1}$. A vector $x \in \mathbb{R}^{n,1}$ is called **time-like**, **light-like**, or **space-like** if $\langle x, x \rangle$ is negative, zero, or positive, respectively.

Hyperbolic n–**space** (as a set) is defined by

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n,1} \mid \langle x, x \rangle = -1,\ x^{n+1} > 0\}.$$

Note that the equation $\langle x, x \rangle = -1$ describes a hyperbeloid with two components, and $\mathbb{H}^n$ is the component with $x^{n+1} > 0$.

Lemma 9.2.1

(i) $\mathbb{H}^n$ *is an* n*-manifold.*

(ii) The tangent space of $\mathbb{H}^n$ *at a point* $x \in \mathbb{H}^n$ *is*

$$T_x\mathbb{H}^n = \{y \in \mathbb{R}^{n,1} \mid \langle x, y \rangle = 0\}.$$

(iii) For any $x \in \mathbb{H}^n$*, the Lorentzian scalar product on* $\mathbb{R}^{n,1}$ *restricts to a positive definite scalar product on* $T_x\mathbb{H}^n$*. This makes* $\mathbb{H}^n$ *a Riemannian manifold.*

Proof. (i) This follows from Theorem 1.3.3 because -1 is a regular value of the function

$$\mathbb{R}^{n+1} \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2.$$

(ii) Apply Proposition 1.4.5.

(iii) This follows from Propositions A.0.1 and A.0.3. □

Recall that the orthogonal group $O(n+1)$ acts on the sphere S^n by isometries. We now introduce a corresponding group for $\mathbb{H}^n$. A linear map $L : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ is called a **Lorentz transformation** if it preserves the Lorentzian scalar product, i.e. if for all $x, y \in \mathbb{R}^{n,1}$ one has

$$\langle Lx, Ly \rangle = \langle x, y \rangle.$$

A real $(n+1) \times (n+1)$ matrix is said to be **Lorentzian** if the corresponding linear map $\mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ is a Lorentz transformation. The **Lorentz group** $O(n,1)$ is the set of all $(n+1) \times (n+1)$ Lorentzian matrices equipped with matrix multiplication.

Every Lorentz transformation $L : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ will map the hyperboloid $\langle x, x \rangle = -1$ to itself. The **orthochronous Lorentz group** $O^+(n,1)$ is the index 2 subgroup of $O(n,1)$ consisting of those elements that map the component $\mathbb{H}^n$ to itself, or equivalently,

$$O^+(n,1) := \{(a_{ij}) \in O(n,1) \,|\, a_{n+1,n+1} > 0\}.$$

Proposition 9.2.1 *For every $A \in O^+(n,1)$ the map*

$$L_A : \mathbb{H}^n \to \mathbb{H}^n, \quad x \mapsto Ax$$

is an isometry.

Proof. This follows immediately from the definitions. □

A Riemannian manifold M is called **frame-homogeneous** if for any two points $p, q \in M$ the following hold. For any orthonormal basis $(v_1, \ldots, v_n)$ for T_pM and orthonormal basis $(w_1, \ldots, w_n)$ for T_qM there is an isometry $\phi : M \to M$ such that $\phi(p) = q$ and $d\phi_p(v_i) = w_i$ for every i.

Theorem 9.2.1 *The Riemannian manifolds $\mathbb{R}^n$, S^n, and $\mathbb{H}^n$ are frame-homogeneous.*

Proof. We give a proof for $\mathbb{H}^n$, leaving the other two cases for the reader. Let $(e_1, \ldots, e_{n+1})$ be the standard basis for R^{n+1}. Then $e_{n+1} \in \mathbb{H}^n$, and $(e_1, \ldots, e_n)$ is an orthonormal basis for $T_{e_{n+1}}\mathbb{H}^n$. It suffices to prove that if $p \in \mathbb{H}^n$ and $(v_1, \ldots, v_n)$ is an orthonormal basis for T_pM then there is an $A \in O^+(n,1)$ such that $Ae_i = v_i$ for $i = 1, \ldots, n$ and $Ae_{n+1} = p$. But this just means that A is the $(n+1) \times (n+1)$ matrix with columns $v_1, \ldots, v_n, p$. Clearly, $A \in O^+(n,1)$, proving that $\mathbb{H}^n$ is frame-homogeneous. □

Conversely, one can show that for $n \geq 2$, any connected, simply-connected, frame-homogeneous Riemannian n–manifold is isometric to either $\mathbb{R}^n$, S^n, or $\mathbb{H}^n$, see [11, 10].

We now define the covariant derivative on $\mathbb{H}^n$. For any $p \in \mathbb{H}^n$ let

$$\Pi_p : \mathbb{R}^{n,1} \to T_pM$$

be the orthogonal projection with respect to the Lorentzian scalar product, i.e.

$$\Pi_p(v) := v + \langle v, p\rangle p$$

for $v \in \mathbb{R}^{n,1}$. Given an open subset $V \subset \mathbb{H}^n$ and a function $f : V \to \mathbb{R}^{n,1}$, we define the tangential part of f as before to be the vector field $f^{\tan}$ on V defined by

$$f^{\tan}(p) := \Pi_p(f(p)).$$

If X, Y are smooth vector fields on V then the covariant derivative $\nabla_X Y$ is by definition the smooth vector field on V defined by

$$\nabla_X Y := (\partial_X Y)^{\tan}.$$

This covariant derivative satisfies the properties stated in Propositions 4.3.2 and 4.3.3. Given a coordinate frame on an open subset $V \subset \mathbb{H}^n$ we define the Christoffel symbols as before by (4.4). Then the proof of Proposition 4.3.5 holds in the current setting as well, expressing the Christoffel symbols in terms of the components of the first fundamental form.

Given a smooth vector field W along a smooth curve $\gamma : I \to \mathbb{H}^n$ we define the covariant derivative $\frac{\nabla}{dt}W$ along γ defined by

$$\frac{\nabla}{dt}W(t) := \Pi_{\gamma(t)}\dot{W}(t).$$

The curve γ is called a geodesic if $\frac{\nabla}{dt}\dot{\gamma}(t) = 0$ for all $t \in I$. If γ takes values in a coordinate domain then γ is a geodesic if and only if the system (6.3) of ordinary differential equations is satisfied. Therefore, the existence and uniqueness results of Theorem 6.2.1 carry over to $\mathbb{H}^n$.

Proposition 9.2.2 *For any $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$ satisfying $\|v\| = 1$ the curve*

$$\alpha : \mathbb{R} \to \mathbb{H}^n, \quad t \mapsto \cosh t \cdot p + \sinh t \cdot v$$

is the maximal geodesic in $\mathbb{H}^n$ starting at p.

Proof. We have

$$\ddot{\alpha}(t) = \cosh t \cdot p + \sinh t \cdot v = \alpha(t),$$

hence $\frac{\nabla}{dt}\dot{\alpha}(t) = 0$. $\square$

We define the exponential map on $\mathbb{H}^n$ in terms of geodesics as before. For any $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$ satisfying $\|v\| = 1$ and any real number t we then have

$$\exp_p(tv) = \cosh t \cdot p + \sinh t \cdot v. \tag{9.2}$$

The traces of the maximal geodesics in $\mathbb{H}^n$ have a simple description which we now explain. If $E \subset \mathbb{R}^{n,1}$ is any 2–dimensional linear subspace which contains time-like vectors then the intersection $\mathbb{H}^n \cap E$ is called a **line** in $\mathbb{H}^n$.

Proposition 9.2.3 *If x, y are distinct points in $\mathbb{H}^n$ then there is a unique line in $\mathbb{H}^n$ passing through x and y.*

Proof. The vectors x and y must be linearly independent, hence span a 2–dimensional subspace $E \subset \mathbb{R}^{n,1}$. Then $\mathbb{H}^n \cap E$ is the unique line passing through x and y. $\square$

Proposition 9.2.4 *The lines in $\mathbb{H}^n$ are exactly the traces of the maximal geodesics.*

Proof. In the xy-plane, the trace of the curve $t \mapsto (\sinh t, \cosh t)$ is the component of the hyperbola $x^2 - y^2 = -1$ where $y > 0$. Combining this with Theorem 9.2.1 we obtain the proposition. $\square$

Lemma 9.2.2 *For any $p \in \mathbb{H}^n$ the exponential map $\exp_p : T_p\mathbb{H}^n \to \mathbb{H}^n$ is a bijection.*

Proof. Surjectivity follows from Propositions 9.2.3 and 9.2.4. To prove injectivity, suppose $w, w' \in T_p\mathbb{H}^n$ and

$$\exp_p(w) = \exp_p(w').$$

We can write

$$w = tv, \quad w' = t'v'$$

where $t, t' \geq 0$ and v, v' are tangent vectors of length 1. Then from Equation 9.2 we get

$$-\cosh t = \langle \exp_p(w), p\rangle = \langle \exp_p(w'), p\rangle = -\cosh t',$$

hence $t = t'$. If $t = 0$ then this yields $w = w' = 0$. Now suppose $t > 0$. From the equation

$$\sinh t \cdot v = \sinh t \cdot v'$$

we then deduce that $v = v'$, so $w = w'$ in this case, too. $\square$

We can now complete the description of the isometry group of $\mathbb{H}^n$.

Theorem 9.2.2 *The map*

$$\tau : O^+(n,1) \to Isom(\mathbb{H}^n), \quad A \mapsto L_A$$

is a group isomomorphism, where L_A is as in Proposition 9.2.1.

Proof. The map τ is obviously a group homomorphism. It is easy to see that the linear span of $\mathbb{H}^n$ is all of $\mathbb{R}^{n+1}$, hence τ is injective. To see that τ is surjective, let $\phi \in \mathrm{Isom}(\mathbb{H}^n)$. Let $(e_1, \ldots, e_{n+1})$ be the standard basis for $\mathbb{R}^{n+1}$, and let $p := e_{n+1}$. Then

$$(d\phi_p(e_1), \ldots, (d\phi_p(e_n)$$

is an orthonormal basis for the tangent space $T_{\phi(p)}\mathbb{H}^n$. As in the proof of Theorem 9.2.1 we see that there is an $A \in O^+(n,1)$ such that the isometries ϕ and L_A of $\mathbb{H}^n$ have the same value and differential at p, i.e.

$$\phi(p) = Ap, \quad d\phi_p = d(L_A)_p.$$

Let $\psi := L_A \circ \phi^{-1}$, so that

$$\psi(p) = p, \quad d\psi_p = \mathrm{Id}.$$

We will show that ψ is the identity. Let $q \in \mathbb{H}^n$. To see that $\psi(q) = q$, recall that by Lemma 9.2.2 there is a maximal geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n$ starting at p such that $\gamma(1) = q$. Then $\beta := \psi \circ \gamma$ is a geodesic in $\mathbb{H}^n$ satisfying

$$\beta(0) = \gamma(0), \quad \beta'(0) = \gamma'(0).$$

The uniqueness result for geodesics, Theorem 6.2.1 (ii), still holds in the current setting, hence we have $\beta = \gamma$. In particular,

$$\psi(q) = \psi(\gamma(1)) = \beta(1) = \gamma(1) = q. \qquad \square$$

We define the Riemannian curvature tensor of $\mathbb{H}^n$ in terms of the covariant derivative as in Section 5.2, and we define the Gauss curvature of the **hyperbolic plane** $\mathbb{H}^2$ in terms of the curvature tensor by Equation (5.1).

Theorem 9.2.3 *The hyperbolic plane has Gauss curvature $K = -1$.*

Proof. We use geodesic polar coordinates. Given $p \in \mathbb{H}^2$, choose an orthonormal basis (v_1, v_2) for $T_p\mathbb{H}^2$ and define

$$h : \mathbb{R}^2 \to \mathbb{H}^2, \quad (r, \theta) \mapsto \exp_p(r\cos\theta \cdot v_1 + r\sin\theta \cdot v_2).$$

Then

$$\left\|\frac{\partial h}{\partial r}\right\|^2 = 1, \quad \left\|\frac{\partial h}{\partial \theta}\right\|^2 = \sinh^2 r, \quad \left\langle \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta} \right\rangle = 0.$$

At any point (r, θ) with $r \neq 0$, the partial derivatives $\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}$ are therefore linearly independent, so the differential $dh_{(r,\theta)}$ is an isomorphism. Thus, h restricts to a surjective local diffeomorphism $\mathbb{R}^2 - \{0\} \to \mathbb{H}^2 - \{p\}$. The calcuation of Lemma 6.6.2 now shows that $K = -1$ on $\mathbb{H}^2 - \{p\}$. By continuity, $K = -1$ on all of $\mathbb{H}^2$. $\square$

Proposition 9.2.5 *Any surface $S \subset \mathbb{R}^3$ of Gauss curvature -1 is locally isometric to the hyperbolic plane. More precisely, for any points $p \in S$ and $q \in \mathbb{H}^2$ there exist neighbourhoods $V \subset S$ of p and $W \subset \mathbb{H}^2$ of q and an isometry $V \to W$.*

Proof. This is proved in the same way as Theorem 6.6.1, using Theorem 5.3.2 and the calculation in the proof of Theorem 9.2.3. $\square$

9.3 Hyperbolic trigonometry

[To be written]

9.4 Conformal maps

[To be written]

9.5 Other models of the hyperbolic space

[To be written]

Problems

9.1 Show that if L_1, L_2 are lines in $\mathbb{H}^2$ and $p_i \in L_i$, $i = 1, 2$, then there is an isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\phi(L_1) = L_2$ and $\phi(p_1) = p_2$.

Appendix A

Symmetric bilinear forms

Let V be an n–dimensional real vector space. By a **bilinear form** on V we mean a bilinear map

$$B : V \times V \to \mathbb{R}.$$

The form is called **symmetric** if

$$B(x, y) = B(y, x)$$

for all $x, y \in V$. The associated **quadratic form**

$$Q : V \to \mathbb{R}, \quad x \mapsto B(x, x)$$

determines B through the polarization identity

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)).$$

The forms B and Q are called **positive definite** if $Q(x) > 0$ for all non-zero $x \in V$. A positive definite symmetric bilinear form is called an **inner product**. If $W \subset V$ is a linear subspace then the restriction of B to $W \times W$ is a bilinear form on W which we denote by $B|_W$.

A symmetric bilinear form B on V is called **non-degenerate** if the linear map

$$V \to V^*, \quad x \mapsto B(x, \cdot)$$

is an isomorphism, where $B(x, \cdot)$ is the element of the dual space V^* given by $y \mapsto B(x, y)$. A non-degenerate symmetric bilinear form B on V is called a **scalar product** on V. It is easy to see that every inner product is a scalar product.

Given a symmetric bilinear form B on V and a subset $S \subset V$, the **orthogonal complement** of S is defined by

$$S^{\perp} := \{x \in V \mid B(x, y) = 0 \text{ for all } y \in S\}.$$

For $x \in V$, the orthogonal complement of the singleton $\{x\}$ is denoted by $x^{\perp}$.

Given linear subspaces W_1, W_2 of V we say that V is the **direct sum** of W_1 and W_2, written

$$V = W_1 \oplus W_2,$$

if every element $x \in V$ can be written uniquely as $x = y_1 + y_2$ where $y_i \in W_i$.

Proposition A.0.1 *Let B be a symmetric bilinear form on V and let $W \subset V$ be a linear subspace such that $B|_W$ is non-degenerate. Then*

$$V = W \oplus W^{\perp}.$$

Proof. Every $x \in V$ defines an element of the dual space W^* given by $y' \mapsto B(x, y')$. Hence there exists a $y \in W$ such that for every $y' \in W$ one has

$$B(x, y') = B(y, y').$$

Let $z := x - y$. Then $z \in W^{\perp}$, so $x = y + z$ is a decomposition of the desired kind. The non-degeneracy of $B|_W$ also implies that $W \cap W^{\perp} = \{0\}$, which implies the decomposition of x is unique. $\square$

Theorem A.0.1 *Let B be a symmetric bilinear form on V and Q the associated quadratic form. Then there exist non-negative integers r, s and a basis $(e_1, \ldots, e_n)$ for V such that for any $(x^1, \ldots, x^n) \in \mathbb{R}^n$ one has*

$$Q\left(\sum_{i=1}^{n} x^i e_i\right) = (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^{r+s})^2.$$

Proof. If $Q \equiv 0$ the theorem is trivial, so suppose $Q \not\equiv 0$. Choose any element $x \in V$ with $Q(x) \neq 0$. By rescaling x we obtain an element $e_1 \in V$ such that $Q(e_1) = \pm 1$. If $n = 1$ then $\{e_1\}$ is a basis for V of the required kind. We now use induction on n. Suppose $n > 2$ and that the theorem holds for all lower values of n. Since $B|_{\mathbb{R}e_1}$ is non-degenerate we have

$$V = \mathbb{R}e_1 \oplus e_1^{\perp}$$

by Proposition A.0.1. Let $(e_2, \ldots, e_n)$ be a basis for $e_1^\perp$ as supplied by the theorem. Then $(e_1, \ldots, e_n)$ is a basis for V. After perhaps reordering the elements of the basis, we obtain a basis for V of the required kind. $\square$

The pair (r, s) is called the **signature** of B. The following result shows that (r, s) is independent of the choice of basis of V.

Proposition A.0.2 (Sylvester's Law of Inertia) *In the situation of Theorem A.0.1, the integer r is the maximal dimension of any linear subspace on which B is positive definite. Similarly, s is the maximal dimension of any linear subspace on which B is negative definite.*

Proof. We prove the assertion about r, the case of s being similar. Clearly, B is positive definite on the linear span of $e_1, \ldots, e_p$. Now suppose W is any linear subspace of V on which B is positive definite. Let Z be the linear span of $e_{p+1}, \ldots, e_n$. Since $Q(x) \leq 0$ for every $x \in Z$ we must have $W \cap Z = \{0\}$. Therefore,

$$\dim W + \dim Z \leq \dim V,$$

or equivalently,

$$\dim W \leq \dim V - \dim Z = p. \qquad \square$$

Proposition A.0.3 *Suppose the vector space V is the direct sum of two subspaces, say*

$$V = W_1 \oplus W_2.$$

Let B be a symmetric bilinear form on V such that $B(x_1, x_2) = 0$ for $x_i \in W_i$. If $B|_{W_i}$ has signature (r_i, s_i) then B has signature $(r_1 + r_2, s_1 + s_2)$.

Proof. Left for the reader. $\square$

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