

MAT4510, Fall 2025

Solutions to assignment.

All references apply to the lecture notes.

Problem 1 Let M and N be manifolds with $M \subset N$, and let $R : N \rightarrow M$ be a smooth retraction. Show that for all $p \in M$ and $v \in T_p M$ one has $dR_p(v) = v$.

Solution: Let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ be a smooth curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then $R(\gamma(t)) = \gamma(t)$ for all t , so by the chain rule we have

$$dR_p(v) = \left. \frac{d}{dt} \right|_0 R(\gamma(t)) = \dot{\gamma}(0) = v.$$

Problem 2 Let S be a regular surface with Gauss curvature K and Riemannian curvature tensor R . Suppose $K = 0$. Show that $R = 0$.

Solution: Let $p \in S$. Choose an orthonormal basis v_1, v_2 for the tangent space $T_p S$. By Theorem 5.3.2 we have

$$\langle R_p(v_1, v_2)v_1, v_2 \rangle = -K(p) = 0.$$

Now let w_1, w_2, z_1, z_2 be arbitrary elements of $T_p S$. We can express

$$w_j = \sum_i a_j^i v_i, \quad z_j = \sum_i b_j^i v_i$$

for some 2×2 real matrices $A = (a_j^i)$ and $B = (b_j^i)$. Because the expression $\langle R_p(w_1, w_2)z_1, z_2 \rangle$ is skew-symmetric both in w_1, w_2 and in z_1, z_2 we have

$$\langle R_p(w_1, w_2)z_1, z_2 \rangle = \det A \cdot \det B \cdot \langle R_p(v_1, v_2)v_1, v_2 \rangle = 0.$$

It follows that $R = 0$.

Problem 3 Let S be a regular surface and $c : I \rightarrow S$ a smooth curve of unit speed, where I is an interval. Suppose N is a smooth vector field on S along c such that for all $t \in I$ one has

$$\|N(t)\| = 1, \quad \langle \dot{c}(t), N(t) \rangle = 0.$$

Prove the following.

(a) There is a smooth function $\kappa : I \rightarrow \mathbb{R}$ such that

$$\frac{\nabla \dot{c}}{dt} = \kappa N, \quad \frac{\nabla N}{dt} = -\kappa \dot{c}.$$

Solution: We first observe that $\dot{c}(t), N(t)$ is an orthonormal basis for $T_{c(t)}S$ for every t . Because

$$0 = \frac{d}{dt} \|\dot{c}(t)\|^2 = \left\langle \frac{\nabla}{dt} \dot{c}(t), \dot{c}(t) \right\rangle,$$

it follows that $\frac{\nabla}{dt} \dot{c}(t)$ must be a real multiple of $N(t)$ for every t . In other words,

$$\frac{\nabla}{dt} \dot{c}(t) = \kappa(t) N(t)$$

where κ is the smooth function given by

$$\kappa(t) = \left\langle \frac{\nabla}{dt} \dot{c}(t), N(t) \right\rangle.$$

Similarly,

$$\frac{\nabla}{dt} N(t) = \bar{\kappa}(t) \dot{c}(t)$$

for some function $\bar{\kappa} : I \rightarrow \mathbb{R}$. But

$$0 = \frac{d}{dt} \langle \dot{c}(t), N(t) \rangle = \left\langle \frac{\nabla}{dt} \dot{c}(t), N(t) \right\rangle + \left\langle \dot{c}(t), \frac{\nabla}{dt} N(t) \right\rangle = \kappa(t) + \bar{\kappa}(t),$$

so $\bar{\kappa} = -\kappa$.

(b) Let $\phi : I \rightarrow \mathbb{R}$ be a primitive of $-\kappa$, i.e. $\dot{\phi} = -\kappa$. Then

$$W(t) := \cos \phi(t) \cdot \dot{c}(t) + \sin \phi(t) \cdot N(t)$$

is a parallel vector field on S along c .

Solution: We have

$$\begin{aligned} \frac{\nabla}{dt} W(t) &= -\dot{\phi}(t) \sin \phi(t) \cdot \dot{c}(t) + \cos \phi(t) \cdot \kappa(t) N(t) \\ &\quad + \dot{\phi}(t) \cos \phi(t) \cdot N(t) - \sin \phi(t) \cdot \kappa(t) \dot{c}(t) \\ &= 0. \end{aligned}$$

Problem 4 Let

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1\}.$$

(a) Show that S is a surface of revolution.

Solution: Let

$$C := \{(x, z) \in \mathbb{R}^2 \mid x^2 = z^2 + 1, x > 0\},$$

and let $\bar{S}$ be the surface of revolution obtained by rotating C about the z -axis. By Lemma 3.4.1, the surface $\bar{S}$ is the image of the map

$$C \times S^1 \rightarrow \mathbb{R}^3, \quad ((x, z), (a, b)) \mapsto (ax, bx, z).$$

Here,

$$(ax)^2 + (bx)^2 = x^2 = z^2 + 1,$$

hence $\bar{S} \subset S$. Conversely, any point in S can be expressed in cylindrical coordinates as $(r \cos \theta, r \sin \theta, z)$ where $r > 0$ and $r^2 = z^2 + 1$, hence $S \subset \bar{S}$. Altogether, this shows that $S = \bar{S}$.

(b) Calculate the Gauss curvature of S .

Solution: We parametrize the curve C by

$$\gamma : \mathbb{R} \rightarrow C, \quad \gamma(t) = (\cosh t, \sinh t).$$

Let

$$F(t, \theta) = (\cosh t \cdot \cos \theta, \cosh t \cdot \sin \theta, \sinh t)$$

be a corresponding local parametrization of S as in formula (3.2). Then (3.5) yields

$$K(F(t, \theta)) = \frac{\cosh t \cdot (\sinh^2 t - \cosh^2 t)}{\cosh t \cdot (\sinh^2 t + \cosh^2 t)^2} = -\frac{1}{(\cosh^2 t + \sinh^2 t)^2}.$$

At any point $(x, y, z) \in S$ we therefore have

$$K(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^2}.$$

(c) Calculate the parallel transport along the curve

$$c : [0, 2\pi] \rightarrow S, \quad c(\theta) = (\sqrt{2} \cos \theta, \sqrt{2} \sin \theta, 1)$$

using the results of Problem 3.

Solution: The curve c does not have unit speed, so we redefine it as

$$c : [0, 2\sqrt{2}\pi] \rightarrow S, \quad c(\theta) = \left(\sqrt{2} \cos \frac{\theta}{\sqrt{2}}, \sqrt{2} \sin \frac{\theta}{\sqrt{2}}, 1 \right),$$

which has unit speed and the same parallel transport as the original curve. We calculate

$$\begin{aligned} \dot{c}(\theta) &= \left(-\sin \frac{\theta}{\sqrt{2}}, \cos \frac{\theta}{\sqrt{2}}, 0 \right), \\ \ddot{c}(\theta) &= -\frac{1}{\sqrt{2}} \left(\cos \frac{\theta}{\sqrt{2}}, \sin \frac{\theta}{\sqrt{2}}, 0 \right). \end{aligned}$$

We also observe that

$$c(\theta) = F \left(t_0, \frac{\theta}{\sqrt{2}} \right),$$

where $t_0 \in \mathbb{R}$ is the solution to

$$\sinh t_0 = 1.$$

(Note that $\cosh t_0 = \sqrt{2}$.) Now define

$$N(\theta) := \frac{1}{\sqrt{3}} \partial_1 F \left(t_0, \frac{\theta}{\sqrt{2}} \right) = \frac{1}{\sqrt{3}} \left(\cos \frac{\theta}{\sqrt{2}}, \sin \frac{\theta}{\sqrt{2}}, \sqrt{2} \right).$$

The vectors $\dot{c}(\theta)$ and $N(\theta)$ form an orthonormal basis for the tangent space $T_{c(\theta)}S$ for every θ . Because $\dot{c}(\theta) \perp \ddot{c}(\theta)$, the covariant derivative

$$\frac{\nabla}{d\theta} \dot{c}(\theta)$$

is the projection of $\ddot{c}(\theta)$ onto $N(\theta)$, i.e.

$$\frac{\nabla}{d\theta} \dot{c}(\theta) = \langle \ddot{c}(\theta), N(\theta) \rangle \cdot N(\theta) = -\frac{1}{\sqrt{6}} \cdot N(\theta).$$

We orient the tangent space $T_{c(0)}S$ by decreeing that the ordered basis $(\dot{c}(0), N(0))$ is positively oriented. To apply the results of Problem 3, let $\phi(\theta) = \frac{\theta}{\sqrt{6}}$. The parallel transport along c is then the rotation of $T_{c(0)}S$ with angle

$$\phi(2\sqrt{2}\pi) = \frac{2\pi}{\sqrt{3}}.$$