

Elementary differential geometry

MAT4510 — Fall 2026 — Lecture notes

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Abstract

These are lecture notes for MAT4510, Fall 2026. They are only meant to be supplementary to the course book [1]; where the lectures follow [1] very closely the notes may be sparse. Any comments from students are welcome.

Contents

Part I. Manifolds	3
1 Manifolds	3
1.1 Basic Notions	3
1.2 Manifolds	3
1.3 Regular Values	3
1.4 Tangent spaces	3
1.5 Vector fields	5
Part II. Lengths and distances	6
2 Lengths and distances	6
2.1 Inner products	6
2.2 The first fundamental form	6
2.3 Isometries	7
2.4 Parametrized curves	8
2.5 Some (linear) algebra	9
Part III. Gaussian curvature and the Theorema Egregium	11
3 Starting from curves and simpler examples	11
3.1 What the theorem says	11
3.2 Classical surface theory: how a surface curves	12
3.3 Curves in a plane	12
3.4 Some graphs in $\mathbb{R}^3$	13
3.5 From the first fundamental form	15
3.6 A more general graph	16
3.7 What remains	17
3.8 Statement	17

3.9	Gaussian curvature	18
3.9.1	Definition	18
3.9.2	A formula for K via a parametrization	18
3.10	The first fundamental form and the key identity	19
3.11	The formula at a point with $v_1(0) = e_1, v_2(0) = e_2$	20
3.12	Proof of the Theorema Egregium	20
Part IV. The covariant derivative		22
4	The covariant derivative	22
4.1	Directional derivatives	22
4.2	The Lie bracket	22
4.3	The covariant derivative	22
Part V. The Riemannian curvature tensor		23
5	The Riemannian curvature tensor	23
5.1	Some algebra	23
5.2	The Riemannian curvature tensor	23
5.3	Theorema Egregium	25
Part VI. Topics still to come		27
6	Geodesics	27
7	Differential forms	27
8	The Gauss-Bonnet Theorem	27
Acknowledgement		27

Part I. Manifolds

1 Manifolds

1.1 Basic Notions

See [1] Section 1.1.

1.2 Manifolds

See [1] Section 1.2. We consider an additional construction here; notation requires considering [1].

Consider a local parametrization $F : U \rightarrow \mathbb{R}^3$ of a regular surface with $U \subset \mathbb{R}^2$. We would like to extend the map F to a local diffeomorphism of a neighbourhood of $U \subset \mathbb{R}^3$ preserving normal directions. The usefulness of this will become apparent later. Define $\tilde{N}$ on U by

$$\hat{N}(x, y) = \text{cross}(\partial_1 F, \partial_2 F) \cdot \|\text{cross}(\partial_1 F, \partial_2 F)\|^{-1} \quad \hat{N} = N \circ F$$

Then N is a unit normal along S . Define $\hat{F}(x, y, z) = F(x, y) + z \cdot \hat{N}(x, y)$. Then the Jacobian determinant of $\hat{F}$ along U is $[\partial_1 \hat{F} : \partial_2 \hat{F} : \hat{N}]$ which is of rank 3 by construction, and it satisfies the special condition that its third column is of unit length and perpendicular to the two first. This construction may be used to prove the following.

Lemma 1.1. *Let S_1, S_2 be regular surfaces in $\mathbb{R}^3$ and assume that N_1, N_2 are unit smooth unit normals along S_1 and S_2 respectively. Assume that $\phi : S_1 \rightarrow S_2$ is a diffeomorphism. Then the map defined by*

$$\phi(p + tN_1(p)) = \phi(p) + tN_2(\phi(p))$$

defines a smooth map on some open neighborhood of S_1 in $\mathbb{R}^3$ into $\mathbb{R}^3$. It is a smooth extension of ϕ and by construction we have that

$$d\phi_p(N_1(p)) = N_2(\phi(p))$$

1.3 Regular Values

See [1] Section 1.3.

1.4 Tangent spaces

The presentation of the tangent space differs slightly from that in [1]. The notion is of course the same.

Let $M \subset \mathbb{R}^n$ be a k -manifold, let $p \in M$, let $\phi : V_p \rightarrow \mathbb{R}^k$ be a chart at p , and write $q = \phi(p)$. Then $F : U_q = \phi(V_p) \rightarrow M$ is a local parametrization at p . The linear space

$$dF_q(\mathbb{R}^k) \subset \mathbb{R}^n$$

is called the *tangent space* of M at p and is denoted by $T_p M$. By definition, the map dF_q is a isomorphisms.

For the moment it might seem that the definition of $T_p M$ as a set depends on the particular chart/parametrization, but the following shows that it does not.

Lemma 1.2. A vector $v \in \mathbb{R}^n$ belongs to $T_p M$ if and only if there exists a smooth curve

$$\gamma : (-\epsilon, \epsilon) \rightarrow M \text{ with } \gamma(0) = p \text{ and } \gamma'(0) =: d_0 \gamma(1) = v$$

Proof. Let $u \in \mathbb{R}^k$. Then $\hat{\gamma}(t) = q + t \cdot u$ is a smooth curve with

$$\hat{\gamma}'(t) = u$$

We have that

$$\gamma'(t) =: \frac{d}{dt} \big|_{t=0} F(\gamma(t)) = dF_0(u) \in T_p(M)$$

This shows that all vectors in $T_p M$ arise of (calculus) derivatives of curves.

On the other hand consider $\gamma : (-\epsilon, \epsilon) \rightarrow M$. Then $\hat{\gamma} = \phi \circ \gamma$ is a curve in $\mathbb{R}^k$ with $\hat{\gamma}(0) = q$, and by the Chain Rule we have that

$$dF_q(\hat{\gamma}'(0)) = \gamma'(0)$$

□

In the sequel will implicitly identify tangent vectors of M at a point p with certain smooth curves in M passing through p . Two curves γ_1 and γ_2 represent the same tangent vector if and only if $\gamma_1'(0) = \gamma_2'(0)$.

Suppose now $H : M \rightarrow N$ is a smooth map between manifolds and let $p \in M$. Since any curve $\gamma : (-\epsilon, \epsilon) \rightarrow M$ with $\gamma(0) = p$ transforms to a curve through $F(p)$ in N by setting $\hat{\gamma}(t) = H(\gamma(t))$ we have that H induces a map

$$dH_p : T_p M \rightarrow T_{F(p)} N$$

The map dH_p is called the *differential* of H at p or the *tangent map* of H at p .

Recall that if ϕ is a local chart on M at p with $F(p) = q$ then

$$dF_q : \mathbb{R}^k \rightarrow T_p M (k = k_M)$$

is an isomorphism. We have that

$$d(H \circ F)_q = dH_p \circ dF_q$$

and we have that dH_p is a linear map.

If $\hat{\phi}$ is a local chart at $F(p)$ we may consider the linear map

$$d(\hat{\phi} \circ H \circ F)_q : \mathbb{R}^k \rightarrow \mathbb{R}^l (l = l_N)$$

which is called a *local representation* of dH_p (depending on ϕ and $\hat{\phi}$).

Proposition 1.3. (Chain Rule)(P1.4.3) Let

$$M \xrightarrow{F} N \xrightarrow{G} P$$

be smooth maps. Then

$$F(G \circ F)_p = dG_q \circ dF_p, p \in M, q = F(p)$$

Theorem 1.4. (Inverse Function Theorem)(T1.4.1) Let $H : M \rightarrow N$ be a smooth map between manifolds. If $p \in M$ is a point where the differential dH_p is an isomorphism then H maps some neighborhood of p in M diffeomorphically onto a neighbourhood of $H(p)$ in N .

Directional derivatives Let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ with $\gamma(0) = p$ represent an element v of $T_p M$. If f is a smooth map near p in M we define

$$\nabla_v f(p) = \frac{d}{dt} \Big|_{t=0} f(\gamma(t))$$

This is called the derivative of f in direction v . If $\hat{f}$ is any map which is smooth near p in $\mathbb{R}^n$ with $\hat{f}|_M = f$ we have that

$$\nabla_v f(p) = d\hat{f}(p)(v)$$

1.5 Vector fields

Let $M \subset \mathbb{R}^n$ be a manifold. A *vector field* X on M is a map

$$X : M \rightarrow \mathbb{R}^3 \text{ with } X(p) \in T_p M \text{ for all } p \in M$$

If f is a smooth function on M and if $p \in M$ we define $\nabla_X f$ as the directional derivative defined above. If Y is a smooth vector field on M we may apply X to consider $\nabla_X Y$. Assuming that Y is smooth on a neighborhood of M in $\mathbb{R}^n$ we have $\nabla_X Y = dY(X)$.

Note that we have not claimed that $\nabla_Y X$ is a vector field on M . In fact this will "typically" not be the case, and the vector $\nabla_Y X$ will point "out of" the manifold M .

Proposition 1.5. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be manifolds and let $f : X \rightarrow Y$ be smooth and let $g : Y \rightarrow \mathbb{R}^k$ be a smooth function. Let $f^*g = g \circ f$. Let $v \in T_p(X)$ and write $v_* \in T_{f(p)}Y$ for $df_p(v)$. Then

$$\nabla_v f^*g = \nabla_{v_*} g$$

Proof. This follows immediately from the definitions representing the tangent vectors by curves. $\square$

Corollary 1.6. Let X be a manifold and let $H : U \rightarrow X$ be a local parametrization with $q = H(p)$. Let g be a smooth function near q . Then

$$\nabla_{\partial_j H} g = \frac{\partial(H^*g)}{\partial u^j}$$

Example 1.7. Let $X : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the map $X(x, y) = (-y, x)$. Letting $\rho(x, y) = x^2 + y^2$ and letting $\nabla \rho$ denote the gradient of ρ we see that

$$\langle \nabla \rho, X \rangle = 0$$

and so X is a vector field on all level sets $c \neq 0$ of ρ . We have that

$$\nabla_X X = dX(X) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \text{ and so } \langle \nabla \rho, \nabla_X X \rangle = -2(x^2 + y^2)$$

and so $\nabla_X X$ is not tangent to any of those level sets.

Part II. Lengths and distances

2 Lengths and distances

2.1 Inner products

On $\mathbb{R}^n$ the Euclidean inner product is defined by

$$\langle u, v \rangle = \sum_{k=1}^n u^k \cdot v^k$$

The *length* or *norm* of a vector u is defined by

$$\|u\| = \sqrt{\langle u, u \rangle} \quad (2.1)$$

and the *distance* between two vectors u and v is $\|u - v\|$. The *angle* θ between two vectors u, v is defined by

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} \quad (2.2)$$

The inner product gives rise to geometry on $\mathbb{R}^n$.

For a vector space V an inner product $\langle \cdot, \cdot \rangle$ on V is a symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

with $\langle v, v \rangle \geq 0$ with equality only if $v = 0$. It follows that the Schwarz inequality holds

$$\langle u, v \rangle^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$$

and it may be used to show that the inner product induces a norm as in (2.1).

2.2 The first fundamental form

(Section 2.1 in [1])

Let $M \subset \mathbb{R}^n$ be a manifold of dimension k . For each $p \in M$ the tangent space $T_p M$ is a k -dimensional vector space, and it inherits an inner-product

$$\langle u, v \rangle_p = g_p(u, v) = \langle u, v \rangle$$

Then map $p \mapsto \langle \cdot, \cdot \rangle_p = g_p$ is called the *first fundamental form* of M and is often denoted by I_p .

If $B = \{v_1, \dots, v_k\}$ is a basis for $T_p M$ any vectors $u, v \in T_p M$ are on the form

$$u = \sum_i a^i v_i \text{ and } v = \sum_j b^j v_j$$

and we have that

$$\langle u, v \rangle = \langle \sum_i a^i v_i, \sum_j b^j v_j \rangle = \sum_{ij} a^i b^j \langle v_i, v_j \rangle$$

and so the inner-product is determined by the numbers $\langle v_i, v_j \rangle$. They make up a symmetric $(k \times k)$ -matrix $I_B = [\langle v_i, v_j \rangle]$, and with respect to matrix multiplication

$$\langle u, v \rangle = a^T I_B b$$

If $F : U \rightarrow M$ is a local parametrization we have the smooth vector fields X_j defined by

$$X_j \circ F = \partial_j F$$

on $V = F(U)$, and the collection $\{X_1(p), \dots, X_n(p)\}$ is a basis for $T_p M$ for each $p \in V$. In this case the inner product is given by the matrix $[g_{ij}]$

$$g_{ij} = \langle X_i, X_j \rangle$$

and we set

$$\tilde{g}_{ij} = g_{ij} \circ F$$

Example 2.1. Consider

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

We may parametrize S locally by

$$F : (0, 2\pi \times \mathbb{R}) \text{ where } F(\theta, h) = (\cos \theta, \sin \theta, h)$$

Then

$$\frac{\partial F}{\partial \theta} = (-\sin \theta, \cos \theta, 0) \text{ and } \frac{\partial F}{\partial h} = (0, 0, 1)$$

In this case we see that

$$[\tilde{g}_{ij}] = \text{Id}$$

The map in Example 2.1 is a local isometry.

Example 2.2. Consider the local parametrization of $\mathbb{R}^2$ given by polar coordinates

$$F : \mathbb{R} \times (0, 2\pi) \rightarrow \mathbb{R}^2 \text{ with } F(r, \theta) = (r \cos \theta, r \sin \theta)$$

We have

$$F_r = (\cos \theta, \sin \theta) \text{ and } F_\theta = (-r \sin \theta, r \cos \theta)$$

So we have that

$$\tilde{g}_{ij} = \begin{bmatrix} \langle F_r, F_r \rangle & \langle F_r, F_\theta \rangle \\ \langle F_\theta, F_r \rangle & \langle F_\theta, F_\theta \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

2.3 Isometries

(Section 2.2 in [1])

Definition 2.3. Let $\phi : M \rightarrow M'$ be a smooth map. We say that ϕ is a *local isometry* if for each $p \in M$ the map

$$dF_p : T_p \rightarrow T_{F(p)} M$$

is an isometry of vector spaces.

The map in Example 2.1 is a local isometry.

2.4 Parametrized curves

(Section 2.3 in [1])

- A *parametrized curve* in M is a smooth map $\gamma : I \rightarrow M$
- The curve γ is *regular* if $\gamma'(t) \neq 0$ for all t
- The image $[\gamma] = \gamma(I)$, also called the *trace* of γ is a manifold.
- The vector $v(t) = \gamma'(t)$ is called the *velocity* of γ , and the scalar $s(t) = \|v(t)\|$ is called the *speed* of γ . If $s \equiv 1$ we say that γ has *unit speed*.
- If J is another interval and if $u : J \rightarrow I$ is a diffeomorphism then $\gamma \circ u$ is called a *reparametrization*.
- If γ is a regular curve then its *length* is defined by

$$L(\gamma) = \int_a^b s(t) dt$$

Lemma 2.4. (Lemma 2.3.1) Every regular curve γ has a unit speed forward reparametrization.

Proof. We are looking for a diffeomorphism $u : I \rightarrow I$ such that the reparametrization $\tilde{\gamma}(t) = \gamma(u(t))$ satisfies $\tilde{s}(t) = 1$. Let $v'(t)$ satisfy $v'(t) = |\gamma'(t)|$ (the fundamental theorem of analysis) and set $u = v^{-1} : v(I) \rightarrow I$. We have that

$$\tilde{v}(t) = \gamma'(u(t)) \cdot u'(t) = \gamma'(u(t))v'(u(t))^{-1} = 1$$

□

Important reformulation: For any regular curve $\gamma : I \rightarrow \mathbb{R}^n$ there exist an interval J and a unit speed reparametrization $\tilde{\gamma} : J \rightarrow [\gamma]$, which is a local isometry. It follows that any two 1-manifolds in $\mathbb{R}^n$ are *locally* isometric.

Lemma 2.5. Every reparametrization of a regular curve γ has the same length as γ .

Proof. This is the chain rule. □

Example 2.6. Consider a curve

$$\gamma_r(t) = (r \cos \theta, r \sin \theta)$$

parametrized by some interval $[0, a]$. Then

$$L[\gamma_r] = \int_0^a \|(-r \sin \theta, r \cos \theta)\| dt = \int_0^a r dt = ar$$

Example 2.7. Suppose we introduce on $\mathbb{R}^2$ the metric from Example 2.2 given by

$$\langle u, v \rangle_{(x,y)} = (u^1, u^2)^T \begin{bmatrix} 1 & 0 \\ 0 & x^2 \end{bmatrix} (v^1, v^2) = u^1 \cdot v^1 + x^2 \cdot u^2 \cdot v^2$$

With respect to this metric we would define the length of a curve γ by

$$L[\gamma] = \int_0^a \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}^{1/2} dt$$

If γ were the vertical curve

$$\gamma(t) = (x, t), t \in [0, a]$$

we would get

$$L[\gamma] = \int_0^a \|(0, 1)\|_{(x,t)}^{1/2} dt = \int_0^a x dt = ax$$

2.5 Some (linear) algebra

For an $(m \times n)$ -matrix $A = [a_i^j]$ we let a_i denote the i th row vector and a^j denote the j th column vector of A . The *matrix product* of an $(m \times n)$ matrix A and an $(n \times p)$ B is the $(m \times p)$ -matrix

$$AB = [(AB)_i^j] = [\langle a_i, b^j \rangle]$$

and it is associative. The *transpose* A^T of a matrix $A = [a_i^j]$ is the matrix $A^T = [a_j^i]$. We have that $(AB)^T = B^T A^T$. If we think of vectors in $\mathbb{R}^n$ as $(n \times 1)$ -vectors the Euclidean inner product may be written by matrix multiplication as

$$\langle u, v \rangle = u^T v$$

and if A is an $(n \times n)$ -matrix we have that

$$\langle Au, v \rangle = (Au)^T v = u^T A^T v = u^T (A^T v) = \langle u, A^T v \rangle$$

In the case that $n = 2$ the *determinant* of a matrix A is defined by

$$\det(A) = a_1^1 \cdot a_2^2 - a_1^2 \cdot a_2^1$$

The quantity $\det(A)$ measures the signed area of the parallelogram spanned by its columns. It may be verified that

$$\det(A^T) = \det(A) \quad \text{and} \quad \det(AB) = \det(A) \cdot \det(B) \quad \text{for all } A, B \quad (2.3)$$

If for an $(n \times n)$ -matrix $A = [a_i^j]$ we let $A(i)$ denote the $(n-1) \times (n-1)$ -matrix obtained by deleting the first column and i th row from A the determinant of a general matrix A may be defined by induction

$$\det(A) = a_1^1 \cdot \det A(1) - a_2^1 \cdot \det(A(2)) + \cdots + (-1)^{n+1} a_n^1 \cdot \det A(n)$$

Again it signifies the volume of the parallelepiped spanned by the columns of A , and it may be verified that the relations (2.3) hold.

Lemma 2.8. For $A = [v_i^j] \in M(n)$

$$\det[\langle v_i, v_j \rangle] = \det[\langle v^i, v^j \rangle] = (\det A)^2$$

Proof. We have that $A^T = [a_i^j]$ with $a_i^j = v_j^i$. So

$$AA^T = [u_i^j][a_i^j] = [\langle u_i, a^j \rangle] = [\langle v_i, v_j \rangle]$$

So

$$\det[\langle v_i, v_j \rangle] = \det(AA^T) = \det(A) \cdot \det(A^T) = \det(A) \cdot \det(A)$$

□

Bilinear maps on $\mathbb{R}^n$. Suppose $W : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map. Then

$$\Lambda(u, v) = \langle Wu, v \rangle \text{ defined for } u, v \in \mathbb{R}^n \quad (2.4)$$

is a bilinear map (on the other hand, any bilinear map is on this form). If $B = \{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ we have that both W and Λ may be represented by matrices W_B and Λ_B with respect to B , i.e., if

$$u = \sum_i a^i v_i \text{ and } v = \sum_i b^i v_i$$

then

$$[Wu]_B = W_B a \text{ and } \langle u, v \rangle = b^T \Lambda_B a$$

Lemma 2.9. Define $A = [v_1 : \dots : v_n]$. We have that

$$W_B = (A^{-1})^T W_{St} A \text{ and } \Lambda_B = A^T W_{St} A$$

and we have that

$$\det W_{St} = \det W_B \text{ and } \det(\Lambda_B) = \det(A)^2 \cdot \det W_{St} = \det[\langle v_i, v_j \rangle] \cdot \det W_{St}$$

and summed up

$$\det(\Lambda_B) = \det[\langle v_i, v_j \rangle] \cdot \det W \quad (2.5)$$

Proof. We consider Λ_B , and we write $W = W_{St}$. Note that $A = [v_j^i]$. Write

$$u = \sum_i a^i v_i = Aa \text{ and } v = \sum_i b^i v_i = Ab$$

We have that

$$\langle Wu, v \rangle = \langle WAa, Ab \rangle = \langle A^T WAa, b \rangle = b^T (A^T WA) a$$

□

The following gives the analogue in a more general setting.

Let V be a vector space. An *inner product* on V is a symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

such that

$$\langle u, u \rangle \geq 0 \text{ if and only if } u = 0 \quad (2.6)$$

The *Cauchy-Schwarz-inequality* tells us that

$$\langle u, v \rangle \leq \sqrt{\langle u, u \rangle} \cdot \sqrt{\langle v, v \rangle}$$

and the inner product gives rise to a *norm* $\|u\| = \sqrt{\langle u, u \rangle}$ on V as in (2.1). The angle between vectors may be defined as in (2.2).

For a basis $B = \{v_1, \dots, v_n\}$ we may define $S = S_B = [\langle v_i, v_j \rangle]$, and with respect to matrix multiplication we get that

$$\langle u, v \rangle = u_B^t S v_B = u^t S v$$

where the middle expression means matrix multiplication, where u_B, v_B are the vectors u, v are represented with respect to the basis B , and the right hand expression is abuse of notation

to avoid the subscript B when a choice of basis is implicit. The condition (2.6) implies that S is invertible.

Recall that if $W : V \rightarrow V$ is a linear map, then $\det(W)$ is well defined. Representing W as a matrix W_B with respect to a basis B , one has that $\det(W_B)$ is independent of the choice of basis (because of the change of basis formula).

Proposition 2.10. *Let V be an n -dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$, and let $\Lambda : V \times V \rightarrow \mathbb{R}$ be a bilinear map. Then there is a unique linear map $W : V \rightarrow V$ such that*

$$\Lambda(u, v) = \langle Wu, v \rangle \text{ for } u, v \in V \quad (2.7)$$

and for any basis $B = \{v_1, \dots, v_n\}$ for V we have that

$$\det(W) \cdot \det[\langle v_i, v_j \rangle] = \det[\Lambda(v_i, v_j)]$$

Proof. Note first that if such a linear map W exists, then it is unique. Assume that a basis B is given and set $S = S_B = [\langle v_i, v_j \rangle]$. With respect to this basis the inner product is given by

$$\langle u, v \rangle = u^t S v$$

where the right hand side is matrix multiplication. The matrix S is invertible. With respect to B and matrix multiplication the map Λ is represented by a matrix $\Lambda_B = \Lambda = [\Lambda(v_i, v_j)]$

$$\Lambda(u, v) = u^t \Lambda v$$

and we get that

$$\Lambda(u, v) = u^t \Lambda v = u^t \Lambda S^{-1} S v = [(\Lambda S^{-1})^t u]^t S v = \langle (\Lambda S^{-1})^t u, v \rangle$$

so we set $W_B = (\Lambda S^{-1})^t$, and we see that $\det(W_B) \cdot \det S = \det \Lambda$. □

Part III. Gaussian curvature and the Theorema Egregium

3 Starting from curves and simpler examples

3.1 What the theorem says

Wrap a sheet of paper around a bottle and it fits without creasing. Try the same on a ball and it will not. Every flat map of a piece of the earth distorts distances, and no cartographer's ingenuity will remove the distortion. These are not failures of craftsmanship but consequences of a theorem.

The theorem is Gauss's, from 1827, and he called it *egregium* — remarkable. It concerns a number attached to each point of a surface, measuring how the surface curves there. Gauss defined that number by looking at the surface from outside: as the point moves, the direction perpendicular to the surface swings around, and the rate of swing is the measure. The construction is external through and through, for it uses a direction pointing out of the surface — one that a two-dimensional creature living inside could not even indicate.

The Theorema Egregium says that the creature could nevertheless compute the number. Anyone who can measure lengths and angles *along* the surface, knowing nothing of any surrounding space, can determine it. Curvature looks like a fact about how a surface lies in space, and is really a fact about the surface alone.

Hence the paper and the ball: bending a sheet preserves lengths measured within it, so it preserves curvature, and plane and sphere have different curvature. Riemann drew the larger conclusion — if curvature needs no surrounding space, then neither does geometry.

Section 3.2 computes curvature by hand in the simplest cases, Section 3.7 proves the theorem, and Parts V and VI give the sharp smoothness requirements and a sequel in complex analysis.

3.2 Classical surface theory: how a surface curves

To understand how a surface S curves we will look at how curves inside the surface curve (not yet having defined what the adjective “curve” might mean). Certain curves in S are also contained in affine planes, and we start by considering such curves.

Before considering the general case we will consider several simpler examples, and we start with the even simpler case of curves in the plane.

3.3 Curves in a plane

For a smooth function h near the origin in $\mathbb{R}$, consider in $\mathbb{R}^2$ the curve γ given by

$$\gamma(x) = (x, h(x))$$

near the origin. The second derivative at the origin, $h''(0)$, measures how much the curve *curves* at the origin. We call this number the *curvature* of γ at the origin.

Example 3.1. Consider

$$h(x) = R - \sqrt{R^2 - x^2},$$

so that γ parametrizes a piece of the circle of radius R centered at $(0, R)$. We have

$$h'(x) = x(R^2 - x^2)^{-\frac{1}{2}} \quad \text{and} \quad h''(x) = (R^2 - x^2)^{-\frac{1}{2}} + x^2(R^2 - x^2)^{-\frac{3}{2}},$$

and so $h''(0) = R^{-1}$.

Taylor expanding h at the origin we have

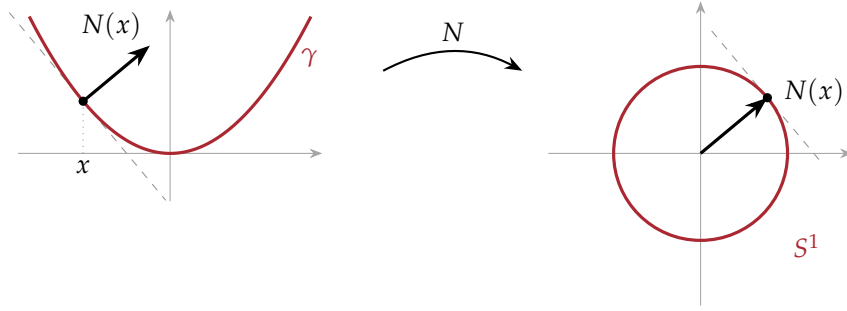
$$h(x) = \frac{1}{2}h''(0)x^2 + a_3x^3 + \cdots,$$

and we see that the curvature only depends on the quadratic part of h at zero. So for the moment, discussing how curves (and soon surfaces) curve, we will consider graphs of quadratic polynomials.

Consider then a curve γ given by $(x, h(x)) = (x, ax^2)$ with $a \in \mathbb{R}$. A way of describing how the curve curves is to describe how a unit normal vector to γ changes. A unit normal vector is given by

$$N(x) = \frac{(-2ax, 1)}{\sqrt{1 + 4a^2x^2}},$$

and N may be thought of as a map from γ (parametrized by the x -axis) into S^1 .



The Gauss map of a plane curve. The two dashed tangent lines are parallel, since both are orthogonal to $N(x)$; this is what lets us read N' as a map of the tangent line of γ to itself.

A computation shows that

$$N'(0) = (-2a, 0) = -h''(0) \cdot (1, 0).$$

So $N'(0)$ is tangent to S^1 at $(0, 1)$, it points along the tangent direction of γ at the origin, and its length is the curvature $h''(0)$ of γ at 0, with a minus sign. This is the point of view — *curvature is the rate of change of the unit normal* — that leads to Definition 3.4 in Section 3.7, which is Gauss's own definition.

3.4 Some graphs in $\mathbb{R}^3$

We consider next a surface S in $\mathbb{R}^3$ which is a graph

$$(x, y, h(x, y)) = (x, y, ax^2 + bxy + cy^2). \quad (3.1)$$

Now the surface may curve differently in different directions. For a line $v(t) = (tv_1, tv_2)$ in $\mathbb{R}^2$ we may consider the curve $t \mapsto (v(t), g_v(t))$ in S , where $g_v(t) = h(v(t))$, and we have

$$g_v''(0) = 2(av_1^2 + bv_1v_2 + cv_2^2) = \langle \mathfrak{H}_h v, v \rangle,$$

where $\mathfrak{H}_h$ is the Hessian

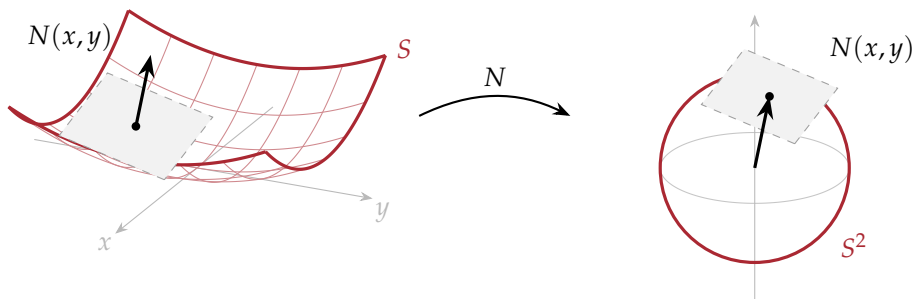
$$\mathfrak{H}_h = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix} (0).$$

We define the *v-curvature* κ_v of S at the origin by

$$\kappa_v \|v\|^2 = \langle \mathfrak{H}_h v, v \rangle.$$

Let us take the other point of view and consider how a unit normal vector to S behaves near the origin. A unit normal vector is given by

$$N(x, y) = \frac{(-2ax - by, -2cy - bx, 1)}{g(x, y)}, \quad g(x, y) = \sqrt{1 + (2ax + by)^2 + (2cy + bx)^2}.$$



The Gauss map of a surface. The tangent plane to S at the marked point and the tangent plane to S^2 at $N(x, y)$ are parallel, both being orthogonal to $N(x, y)$; this is what lets us read $dN(0)$ as a map of the tangent plane to itself, exactly as in the case of a plane curve.

Considering $dN(0)$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and using $g(0) = 1$, $g_x(0) = g_y(0) = 0$, we see that

$$-dN(0) = \mathfrak{H}_h \quad \text{and so} \quad \kappa_v = -\langle dN(0)v, v \rangle.$$

Computation. Write $N = (N^1, N^2, N^3)$ with

$$N^1 = \frac{-h_x}{g}, \quad N^2 = \frac{-h_y}{g}, \quad N^3 = \frac{1}{g}, \quad g = \sqrt{1 + h_x^2 + h_y^2}.$$

Since $h_x(0) = h_y(0) = 0$ we have $g(0) = 1$, and differentiating $g^2 = 1 + h_x^2 + h_y^2$ gives $2gg_x = 2h_x h_{xx} + 2h_y h_{xy}$, so that $g_x(0) = 0$, and likewise $g_y(0) = 0$. Hence, by the quotient rule, at the origin every term containing a derivative of g drops out:

$$N_x^1(0) = \left(\frac{-h_{xx}g + h_x g_x}{g^2} \right)(0) = -h_{xx}(0), \quad N_y^1(0) = -h_{xy}(0),$$

and in the same way $N_x^2(0) = -h_{xy}(0)$, $N_y^2(0) = -h_{yy}(0)$, while $N_x^3(0) = N_y^3(0) = 0$. The tangent plane T_0S is the (x, y) -plane, so reading $dN(0)$ as a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ we have found

$$dN(0) = \begin{pmatrix} -h_{xx} & -h_{xy} \\ -h_{xy} & -h_{yy} \end{pmatrix} (0) = -\mathfrak{H}_h,$$

which is the assertion. Note that the whole computation rests on the single fact that $\nabla h(0) = 0$; this is why the origin is special.

The map $W = -dN(0)$ is the *shape operator* of Section 3.7.

Consider next an arbitrary curve $\gamma(t)$ in $\mathbb{R}^2$ with $\gamma(0) = 0$ and $\gamma'(0) = v \neq 0$. Then

$$\hat{\gamma}(t) = (\gamma(t), h(\gamma(t))) = (\gamma(t), \psi(t))$$

is a curve in S with $\hat{\gamma}'(0) = (v, 0)$, and any regular curve in S near the origin with tangent v has this form. The acceleration vector of $\hat{\gamma}$ is

$$\mathbf{a}(t) = (\gamma''(t), \psi''(t)),$$

and since $\nabla h(0) = 0$ a computation gives $\psi''(0) = \langle \mathfrak{H}_h v, v \rangle$, so that

$$\mathbf{a}(0) = (\cdot, \cdot, \langle \mathfrak{H}_h v, v \rangle) \quad \text{and so} \quad \kappa_v = \mathbf{a}(0) \cdot N(0).$$

In words: the v -curvature is the component of the acceleration along the normal, and it does not depend on which curve with tangent v we chose.

Computation. Write $\gamma = (\gamma_1, \gamma_2)$, so that $\psi = h \circ \gamma$. The chain rule gives

$$\psi'(t) = h_x(\gamma(t)) \gamma_1'(t) + h_y(\gamma(t)) \gamma_2'(t) = \langle \nabla h(\gamma(t)), \gamma'(t) \rangle,$$

and differentiating once more, using the product rule on each of the two terms,

$$\psi''(t) = \langle \mathfrak{H}_h(\gamma(t)) \gamma'(t), \gamma'(t) \rangle + \langle \nabla h(\gamma(t)), \gamma''(t) \rangle.$$

Now evaluate at $t = 0$. We have $\gamma(0) = 0$ and $\gamma'(0) = v$, and $\nabla h(0) = 0$, so the second term vanishes and

$$\psi''(0) = \langle \mathfrak{H}_h v, v \rangle.$$

The second term is the only place where γ'' could have entered, and it is exactly the term killed by $\nabla h(0) = 0$. That is why the answer depends on v alone and not on the curve.

Definition 3.2 (Gaussian curvature; special case of a special graph). The *Gaussian curvature* K of the graph S in (3.1) at the origin is

$$K = \det dN(0) = \det \mathfrak{H}_h = (h_{xx}h_{yy} - h_{xy}^2)(0). \quad (3.2)$$

Note that $\det dN(0) = \det(-\mathfrak{H}_h) = \det \mathfrak{H}_h$, because we are in dimension two; in particular the sign of the chosen normal is irrelevant. This is exactly Definition 3.4 below, in the special situation at hand.

Since $\mathfrak{H}_h$ is a symmetric matrix it has two orthonormal eigenvectors v_1, v_2 with eigenvalues $\kappa_1 \leq \kappa_2$, and $K = \kappa_1 \kappa_2$. The numbers κ_1 and κ_2 are the v_1 - and v_2 -curvatures at the origin; they are called the *principal curvatures* at 0, and the directions v_1 and v_2 are the directions in which S curves the least and the most respectively (signed, depending on a choice of normal).

3.5 From the first fundamental form

Let us consider another way to compute K , still for the graph (3.1). From now on we relax the assumption on h : it need no longer be a quadratic polynomial, but may be any smooth function with $h(0) = 0$ and $\nabla h(0) = 0$. Nothing said so far used more than this — the quadratic case was only a device for seeing which part of h the curvature depends on — and Definition 3.2 and formula (3.2) continue to hold verbatim.

This time we use what is called the *first fundamental form*: writing $X = (1, 0, h_x)$ and $Y = (0, 1, h_y)$ for the coordinate vector fields of the parametrization $(x, y) \mapsto (x, y, h(x, y))$, we set

$$E = \langle X, X \rangle, \quad F = \langle X, Y \rangle, \quad G = \langle Y, Y \rangle.$$

We would like to compute the curvature (3.2) in terms of E, F and G . We have

$$E = 1 + h_x^2, \quad F = h_x h_y, \quad G = 1 + h_y^2,$$

and we check what information we can get from E, F and G by differentiating them:

$$\begin{aligned} E_x &= 2h_x h_{xx}, & E_y &= 2h_x h_{xy}, \\ G_x &= 2h_y h_{xy}, & G_y &= 2h_y h_{yy}, \\ F_x &= h_{xx} h_y + h_x h_{xy}, & F_y &= h_{xy} h_y + h_x h_{yy}. \end{aligned}$$

Since $h_x(0) = h_y(0) = 0$, all six of these vanish at the origin, so first derivatives give us no information at all. Second derivatives do. We have

$$F_{xy} = h_{xxy}h_y + h_{xx}h_{yy} + h_{xy}h_{xy} + h_xh_{xyy} \quad \text{and so} \quad F_{xy}(0) = (h_{xx}h_{yy} + h_{xy}^2)(0),$$

while

$$E_{yy} = 2(h_{xy}h_{xy} + h_xh_{xyy}), \quad G_{xx} = 2(h_{xy}h_{xy} + h_yh_{xxy}),$$

so that

$$E_{yy}(0) = G_{xx}(0) = 2h_{xy}^2(0).$$

Subtracting, the unwanted term h_{xy}^2 cancels and we are left with

$$K = \left[F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) \right](0). \quad (3.3)$$

This is the first appearance of the identity that carries the whole theorem: the left-hand side of (3.3) was defined through the normal vector N , i.e. through the way S sits in $\mathbb{R}^3$, while the right-hand side only involves E, F, G , i.e. lengths of curves *inside* S . Compare (3.10) in Section 3.11: there $E = G = 1$ and $F = 0$ at the origin, and all first derivatives of E, F, G vanish at 0 as well, so the general formula collapses to exactly (3.3).

3.6 A more general graph

Consider now the more general case where $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an invertible linear map and a surface S is parametrized by

$$(x, y) \mapsto (A(x, y), h(x, y)), \quad (3.4)$$

where h is as before. The curvature K is defined as before, but this time S is not immediately given as a graph over the (x, y) -plane in these coordinates. However we may reparametrize S by

$$(x, y) \mapsto (x, y, \hat{h}(x, y)) = (x, y, h(A^{-1}(x, y))), \quad (3.5)$$

and then $K = \det \mathfrak{H}_{\hat{h}}(0)$. A computation shows that

$$\mathfrak{H}_{\hat{h}} = (A^{-1})^T \mathfrak{H}_h A^{-1} \quad \text{and so} \quad K = \det(A)^{-2} \cdot \det \mathfrak{H}_h.$$

Computation. Put $B = A^{-1}$, so that $\hat{h}(w) = h(Bw)$ with B linear, and write $\partial_i = \partial/\partial w_i$. Differentiating once, the chain rule gives

$$\partial_i \hat{h}(w) = \sum_k (\partial_k h)(Bw) B_{ki},$$

since $\partial_i (Bw)_k = B_{ki}$. Differentiating a second time, and using the chain rule again on $(\partial_k h)(Bw)$,

$$\partial_j \partial_i \hat{h}(w) = \sum_{k,l} (\partial_l \partial_k h)(Bw) B_{lj} B_{ki}.$$

The right-hand side is the (i, j) entry of $B^T \mathfrak{H}_h B$, so

$$\mathfrak{H}_{\hat{h}} = B^T \mathfrak{H}_h B = (A^{-1})^T \mathfrak{H}_h A^{-1}.$$

Taking determinants and using $\det B^T = \det B = \det(A)^{-1}$ gives $\det \mathfrak{H}_{\hat{h}} = \det(A)^{-2} \det \mathfrak{H}_h$.

The shape of the answer is worth remembering: the Hessian is not a linear map but a *bilinear form*, and a bilinear form transforms with $B^T(\cdot)B$, not by conjugation. This is exactly why $\det \mathfrak{H}_h$ alone cannot be the curvature here, and why the factor $\det(A)^2 = (EG - F^2)(0)$ has to appear.

Again it is instructive to compute K from the first fundamental form. In this case, with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the surface S is parametrized by

$$B(x, y) = (ax + by, cx + dy, h(x, y)), \quad B_x = (a, c, h_x), \quad B_y = (b, d, h_y),$$

and

$$E = a^2 + c^2 + h_x^2, \quad F = ab + cd + h_x h_y, \quad G = b^2 + d^2 + h_y^2.$$

The constants differentiate away, so exactly as in Section 3.5 we get

$$\left[F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) \right](0) = \det \mathfrak{H}_h,$$

but this time that is *not* equal to the curvature. What is missing is the metric factor: noting again that $h_x(0) = h_y(0) = 0$, a computation shows that

$$(EG - F^2)(0) = (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 = (ad - bc)^2 = \det(A)^2,$$

and so we get

$$K = \left[\left(F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) \right) \cdot (EG - F^2)^{-1} \right](0). \quad (3.6)$$

3.7 What remains

Formula (3.6) is already the Theorema Egregium for the surfaces (3.4): the curvature at the origin has been expressed through E, F, G alone. Two things were special about that situation, and Section 3.7 is devoted to removing them.

First, the parametrizations (3.1) and (3.4) have *affine* first two components, so the vector fields $\hat{v}_1 = (a, c)$ and $\hat{v}_2 = (b, d)$ obtained by projecting B_x, B_y to the (x, y) -plane are constant. For a general parametrization they are not, and their derivatives contribute the extra terms

$$\langle \hat{v}_{1,x}, \hat{v}_{2,y} \rangle - \langle \hat{v}_{1,y}, \hat{v}_{2,x} \rangle$$

appearing in Lemma 3.10; the work consists in showing that these terms, too, can be written in terms of E, F, G (Lemma 3.12), which yields the general formula (3.10).

Second, we only defined K at the origin for a surface tangent to the (x, y) -plane there. For an arbitrary regular surface and an arbitrary point one defines K invariantly as $\det dN$ (Definition 3.4) and checks that a rigid motion of $\mathbb{R}^3$ changes neither K nor E, F, G (Lemma 3.5), so that one may always move the picture into the normalized position used above.

A direct proof of the Theorema Egregium

3.8 Statement

Throughout, a *regular surface* is a smooth embedded surface $S \subset \mathbb{R}^3$, and a local parametrization is a smooth map $H: U \rightarrow S$, $U \subset \mathbb{R}^2$ open, which is a diffeomorphism onto an open subset of S . An *isometry* $\phi: M \rightarrow N$ between regular surfaces is a diffeomorphism with $\langle d\phi_p v, d\phi_p w \rangle = \langle v, w \rangle$ for all $p \in M$ and $v, w \in T_p M$.

Theorem 3.3 (Theorema Egregium). *Let $\phi: M \rightarrow N$ be an isometry between regular surfaces in $\mathbb{R}^3$. Then $K(p) = K(\phi(p))$ for all $p \in M$, where K denotes the Gaussian curvature (Definition 3.4).*

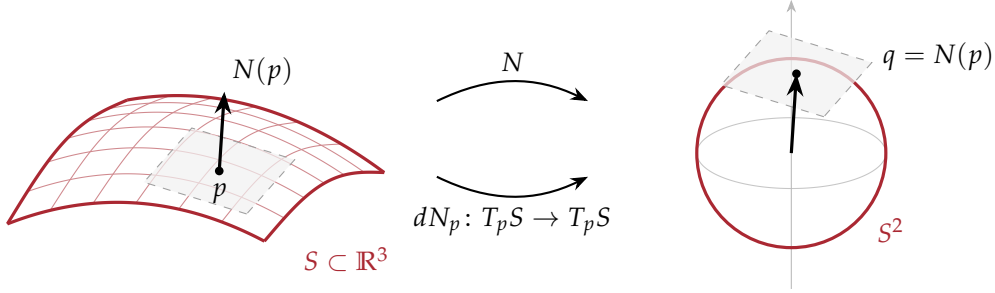
3.9 Gaussian curvature

3.9.1 Definition

Let S be a regular surface, $p \in S$, and let N be a smooth unit normal vector field on a neighbourhood of p in S . Then N is a map into the unit sphere S^2 , and since $T_{N(q)}S^2 = N(q)^\perp = T_qS$, its differential is a linear map

$$dN_q: T_qS \rightarrow T_qS.$$

Definition 3.4. The *Gaussian curvature* of S at p is $K(p) = \det(dN_p)$.



The Gauss map of a general surface $S \subset \mathbb{R}^3$. The point p is sent to $q = N(p) \in S^2$. The dashed tangent planes T_pS and T_qS^2 are parallel, both being orthogonal to $N(p)$, so the differential dN_p is a linear map of the plane T_pS to itself; the Gaussian curvature $K(p)$ is its determinant.

Since $\dim T_pS = 2$, replacing N by $-N$ replaces dN_p by $-dN_p$ and leaves the determinant unchanged; so K is well defined, and $K = \det(dN_p) = \det(-dN_p)$. We write $W = -dN_p$ (the shape operator). For a graph tangent to the (x, y) -plane at p this is Definition 3.2, and $W = \mathfrak{H}_h(0)$.

Lemma 3.5 (Invariance under rigid motions). Let $A(x) = Ux + b$ with $U \in O(3)$, $b \in \mathbb{R}^3$, and let $S' = A(S)$. Then $K_{S'}(A(p)) = K_S(p)$.

Proof. $N' = U \circ N \circ A^{-1}$ is a unit normal field on S' , $T_{A(p)}S' = U(T_pS)$, and by the chain rule $dN'_{A(p)} = U \circ dN_p \circ U^{-1}$ on $T_{A(p)}S'$. Conjugate maps have the same determinant. $\square$

3.9.2 A formula for K via a parametrization

Let $H = (f, g, h): U \rightarrow S$ be a local parametrization with $0 \in U$ and $H(0) = p$, and write

$$v_1 = H_x, \quad v_2 = H_y,$$

vector-valued functions on U whose values at 0 span T_pS . Write $v_j = (v_j^1, v_j^2, v_j^3)$ for the components, and $H_{ij} = \partial^2 H / \partial x_i \partial x_j$ (with $x_1 = x$, $x_2 = y$), so that $H_{ij} = \partial v_i / \partial x_j = \partial v_j / \partial x_i$ is the derivative of v_i with respect to x_j . For later use we note that this is the same as the derivative of v_i in the *direction* v_j , in the sense of Section 1. By Proposition 2.10 of Section 2.5 applied to $\Lambda(u, v) = \langle Wu, v \rangle$ and the basis $\{v_1(0), v_2(0)\}$,

$$K(p) \cdot \det[\langle v_i, v_j \rangle](0) = \det[\langle Wv_i, v_j \rangle](0). \quad (3.7)$$

Lemma 3.6. $\langle Wv_i, v_j \rangle = \langle N, H_{ij} \rangle$ along H ; in particular the matrix $[\langle Wv_i, v_j \rangle]$ is symmetric.

Proof. Differentiate $0 = \langle N \circ H, v_i \rangle$ in the direction x_j :

$$0 = \langle dN(v_j), v_i \rangle + \langle N, H_{ij} \rangle,$$

so $\langle Wv_j, v_i \rangle = -\langle dN(v_j), v_i \rangle = \langle N, H_{ij} \rangle$. Since $H_{ij} = H_{ji}$, this is symmetric in i, j , whence $\langle Wv_i, v_j \rangle = \langle Wv_j, v_i \rangle = \langle N, H_{ij} \rangle$. $\square$

Lemma 3.7. Suppose $v_1^3(0) = v_2^3(0) = 0$, i.e. $T_p S$ is the (x, y) -plane. Then

$$K(p) \cdot \det[\langle v_i, v_j \rangle](0) = \det \mathfrak{H}_h(0),$$

where $\mathfrak{H}_h = [h_{x_i x_j}]$ is the Hessian of the third component h of H .

Proof. Since $N(p)$ is a unit vector orthogonal to $v_1(0), v_2(0)$, which span the (x, y) -plane, $N(p) = \pm e_3$. By Lemma 3.6, $\langle Wv_i, v_j \rangle(0) = \pm \langle e_3, H_{ij}(0) \rangle = \pm h_{x_i x_j}(0)$. The sign drops out of the 2×2 determinant, and (3.7) gives the claim. $\square$

Remark 3.8. One can also see Lemma 3.7 by reparametrizing: if $A = [v_j^i(0)]_{i,j \leq 2}$ and $\hat{H} = H \circ A^{-1}$, then $\hat{H}_x(0) = e_1$, $\hat{H}_y(0) = e_2$, so $K = \det \mathfrak{H}_{\hat{h}}(0)$ directly; and $\mathfrak{H}_{\hat{h}}(0) = (A^{-1})^t \mathfrak{H}_h(0) A^{-1}$ while $\det[\langle v_i, v_j \rangle](0) = (\det A)^2$. This is precisely the computation of Section 3.6, cf. (3.5). The direct argument above avoids the reparametrization.

3.10 The first fundamental form and the key identity

Let $v_1, v_2: U \rightarrow \mathbb{R}^3$ be any smooth vector-valued functions of (x, y) and set

$$E = \langle v_1, v_1 \rangle, \quad F = \langle v_1, v_2 \rangle, \quad G = \langle v_2, v_2 \rangle.$$

When $v_1 = H_x$, $v_2 = H_y$ these are the coefficients of the first fundamental form of S in the parametrization H , as in Section 3.5, and $EG - F^2 = \det[\langle v_i, v_j \rangle]$. In that case also $v_{1,y} = H_{xy} = v_{2,x}$.

Lemma 3.9. If $v_{1,y} = v_{2,x}$, then on all of U

$$F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) = \langle v_{1,x}, v_{2,y} \rangle - \langle v_{1,y}, v_{2,x} \rangle. \quad (3.8)$$

Proof. By the product rule,

$$\begin{aligned} F_{xy} &= \langle v_{1,xy}, v_2 \rangle + \langle v_{1,x}, v_{2,y} \rangle + \langle v_{1,y}, v_{2,x} \rangle + \langle v_1, v_{2,xy} \rangle, \\ \frac{1}{2}E_{yy} &= (\langle v_{1,y}, v_1 \rangle)_y = \langle v_{1,yy}, v_1 \rangle + \langle v_{1,y}, v_{1,y} \rangle = \langle v_{2,xy}, v_1 \rangle + \langle v_{2,x}, v_{1,y} \rangle, \\ \frac{1}{2}G_{xx} &= (\langle v_{2,x}, v_2 \rangle)_x = \langle v_{2,xx}, v_2 \rangle + \langle v_{2,x}, v_{2,x} \rangle = \langle v_{1,xy}, v_2 \rangle + \langle v_{1,y}, v_{2,x} \rangle, \end{aligned}$$

where in the last two lines we used $v_{1,y} = v_{2,x}$ (hence $v_{1,yy} = v_{2,xy}$ and $v_{2,xx} = v_{1,xy}$). Subtracting, the terms $\langle v_{1,xy}, v_2 \rangle$ and $\langle v_1, v_{2,xy} \rangle$ cancel, and

$$F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) = \langle v_{1,x}, v_{2,y} \rangle + \langle v_{1,y}, v_{2,x} \rangle - 2\langle v_{1,y}, v_{2,x} \rangle,$$

which is (3.8). $\square$

Now let $v_1 = H_x$, $v_2 = H_y$ for a parametrization H of S with $H(0) = p$ and $T_p S$ the (x, y) -plane, i.e. $v_j^3(0) = 0$. Write $\hat{v}_j = (v_j^1, v_j^2)$ for the projection to the first two components.

Lemma 3.10. Under these hypotheses, at the point 0,

$$F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) = \langle \hat{v}_{1,x}, \hat{v}_{2,y} \rangle - \langle \hat{v}_{1,y}, \hat{v}_{2,x} \rangle + K(p)(EG - F^2). \quad (3.9)$$

Proof. The third components of $v_{1,x}, v_{2,y}, v_{1,y} = v_{2,x}$ are h_{xx}, h_{yy}, h_{xy} . Hence

$$\langle v_{1,x}, v_{2,y} \rangle - \langle v_{1,y}, v_{2,x} \rangle = \langle \hat{v}_{1,x}, \hat{v}_{2,y} \rangle - \langle \hat{v}_{1,y}, \hat{v}_{2,x} \rangle + (h_{xx}h_{yy} - h_{xy}^2),$$

and $h_{xx}h_{yy} - h_{xy}^2 = \det \mathfrak{H}_h(0) = K(p)(EG - F^2)(0)$ by Lemma 3.7. Insert into (3.8). $\square$

The whole difficulty of the theorem is contained in (3.9): the left-hand side is intrinsic, the last term is K up to a metric factor, and it remains only to see that the two middle terms are intrinsic as well. This is a computation at a single point.

Remark 3.11. For the parametrizations of Section 3.2 the two middle terms vanish identically: there $\hat{v}_1 = (a, c)$ and $\hat{v}_2 = (b, d)$ are constant, and (3.9) reduces to (3.6) — and, when A is the identity, to (3.3).

3.11 The formula at a point with $v_1(0) = e_1, v_2(0) = e_2$

Lemma 3.12. *Suppose in addition that $v_1(0) = e_1$ and $v_2(0) = e_2$. Then at 0*

$$\begin{aligned}\hat{v}_{1,x} &= \left(\frac{1}{2}E_x, F_x - \frac{1}{2}E_y\right), \\ \hat{v}_{2,y} &= \left(F_y - \frac{1}{2}G_x, \frac{1}{2}G_y\right), \\ \hat{v}_{1,y} &= \hat{v}_{2,x} = \left(\frac{1}{2}E_y, \frac{1}{2}G_x\right).\end{aligned}$$

Proof. At 0, since $v_1 = e_1$ and $v_2 = e_2$ there, the product rule gives

$$\begin{aligned}E_x &= 2\langle v_{1,x}, v_1 \rangle = 2v_{1,x}^1, & E_y &= 2v_{1,y}^1, & G_x &= 2v_{2,x}^2, & G_y &= 2v_{2,y}^2, \\ F_x &= \langle v_{1,x}, v_2 \rangle + \langle v_1, v_{2,x} \rangle = v_{1,x}^2 + v_{2,x}^1, & F_y &= v_{1,y}^2 + v_{2,y}^1.\end{aligned}$$

Using $v_{1,y} = v_{2,x}$ we get $v_{2,x}^1 = v_{1,y}^1 = \frac{1}{2}E_y$ and $v_{2,x}^2 = v_{2,x}^1 = \frac{1}{2}G_x$. Hence $v_{1,x}^2 = F_x - \frac{1}{2}E_y$ and $v_{2,y}^1 = F_y - \frac{1}{2}G_x$, and the three displayed formulas follow. $\square$

Proposition 3.13. *Let H be a parametrization of S with $H(0) = p$, $H_x(0) = e_1$, $H_y(0) = e_2$. Then*

$$K(p) = \left[F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) + \frac{1}{4}(E_y^2 + G_x^2 + E_x G_x + E_y G_y) - \frac{1}{2}(E_x F_y + G_y F_x) \right]_{(0)}. \quad (3.10)$$

Proof. At 0 we have $E = G = 1, F = 0$, so $EG - F^2 = 1$, and Lemma 3.10 gives $K(p) = F_{xy} - \frac{1}{2}(E_{yy} + G_{xx}) - \langle \hat{v}_{1,x}, \hat{v}_{2,y} \rangle + \langle \hat{v}_{1,y}, \hat{v}_{2,x} \rangle$. By Lemma 3.12,

$$\begin{aligned}\langle \hat{v}_{1,x}, \hat{v}_{2,y} \rangle &= \frac{1}{2}E_x(F_y - \frac{1}{2}G_x) + (F_x - \frac{1}{2}E_y)\frac{1}{2}G_y = \frac{1}{2}(E_x F_y + G_y F_x) - \frac{1}{4}(E_x G_x + E_y G_y), \\ \langle \hat{v}_{1,y}, \hat{v}_{2,x} \rangle &= \frac{1}{4}(E_y^2 + G_x^2).\end{aligned}$$

Substituting gives (3.10). $\square$

The right-hand side of (3.10) involves only E, F, G and their derivatives up to order two at 0: it is determined by the first fundamental form in the parametrization H . Note that for the graph (3.1) of Section 3.4 all first derivatives of E, F, G vanish at the origin, so that (3.10) is precisely (3.3).

3.12 Proof of the Theorema Egregium

Lemma 3.14. *Let S be a regular surface and $p \in S$. Every parametrization H of S with $H(0) = p$, $H_x(0), H_y(0)$ orthonormal and $T_p S$ the (x, y) -plane can be brought, by a rigid motion A of $\mathbb{R}^3$ fixing p and preserving the (x, y) -plane, to a parametrization $A \circ H$ of $A(S)$ with $(A \circ H)_x(0) = e_1$, $(A \circ H)_y(0) = e_2$. The functions E, F, G of $A \circ H$ coincide with those of H , and $K_{A(S)}(p) = K_S(p)$.*

Proof. $H_x(0), H_y(0)$ is an orthonormal basis of the (x, y) -plane, so there is $U \in O(3)$ fixing e_3 with $UH_x(0) = e_1$, $UH_y(0) = e_2$; take $A(q) = U(q - p) + p$. Since U is orthogonal, $\langle UH_{x_i}, UH_{x_j} \rangle = \langle H_{x_i}, H_{x_j} \rangle$ everywhere, so E, F, G are unchanged. The last claim is Lemma 3.5. $\square$

Proof of Theorem 3.3. Fix $p \in M$. By Lemma 3.5 we may apply rigid motions to M and to N separately without changing curvature or the isometry property; so assume $p = 0$, $T_p M$ is the (x, y) -plane, and likewise $\phi(p) = 0$ and $T_{\phi(p)} N$ is the (x, y) -plane.

Choose a parametrization H of M near p with $H(0) = p$ and $H_x(0) = e_1, H_y(0) = e_2$; for instance, since $T_p M$ is the (x, y) -plane, M is locally the graph of a function h with $h(0) = 0$, $\nabla h(0) = 0$, and $H(x, y) = (x, y, h(x, y))$ will do. Let E, F, G be its first fundamental form. By Proposition 3.13, $K_M(p)$ is given by (3.10) evaluated on E, F, G .

Now $\tilde{H} = \phi \circ H$ is a parametrization of N near $\phi(p)$ with $\tilde{H}(0) = \phi(p)$, and because ϕ is an isometry,

$$\tilde{E} = \langle d\phi(H_x), d\phi(H_x) \rangle = \langle H_x, H_x \rangle = E, \quad \tilde{F} = F, \quad \tilde{G} = G$$

as functions on the parameter domain. In particular $\tilde{H}_x(0), \tilde{H}_y(0)$ are orthonormal; they lie in $T_{\phi(p)}N$, the (x, y) -plane. By Lemma 3.14 there is a rigid motion A such that $A \circ \tilde{H}$ parametrizes $A(N)$ with coordinate vectors e_1, e_2 at 0, with the same E, F, G , and with

$$K_{A(N)}(\phi(p)) = K_N(\phi(p))$$

Proposition 3.13 applied to $A \circ \tilde{H}$ gives $K_N(\phi(p))$ as (3.10) evaluated on the same functions E, F, G . Hence $K_N(\phi(p)) = K_M(p)$. $\square$

Remark 3.15. Formula (3.10) is the Brioschi formula specialized to a point where $E = G = 1$, $F = 0$.

Part IV. The covariant derivative

4 The covariant derivative

Motivation

Recall the formula (3.7)

$$K(p) \cdot \det[\langle v_i, v_j \rangle](0) = \det[\langle Wv_i, v_j \rangle](0) \quad (4.1)$$

which served as our basis for approaching K . Here v_1, v_2 is a pair of linearly independent vector fields on a regular surface S , N denotes a unit normal field along S , and $W = -dN$. The bilinear map

$$II(u, v) = \langle Wu, v \rangle$$

is called the *second fundamental form* of S and the matrix $[\langle Wv_i, v_j \rangle]$ is the representation matrix for II with respect to the frame v_1, v_2 . The *first fundamental form* I is Euclidean inner product on S and it is represented by its metric components

$$E = \langle v_1, v_1 \rangle \quad F = \langle v_1, v_2 \rangle \quad G = \langle v_2, v_2 \rangle \quad (4.2)$$

The essence of Theorema Egregium is to recover the right hand side of (4.1) from (4.2).

We reduced the right hand side of (4.1) to something for us more useful. Noting that $\langle N, v_j \rangle \equiv 0$ we get

$$\begin{aligned} 0 &= \nabla_{v_i} \langle N, v_j \rangle = \langle \nabla_{v_i} N, v_j \rangle + \langle N, \nabla_{v_i} v_j \rangle = \langle dNv_i, v_j \rangle + \langle N, \nabla_{v_i} v_j \rangle \\ &= \langle -Wv_i, v_j \rangle + \langle N, \nabla_{v_i} v_j \rangle \end{aligned}$$

and we worked with $\langle N, \nabla_{v_i} v_j \rangle$; the object $\nabla_{v_i} v_j$ turned out to be crucial. The right hand side of (4.1) now becomes (suppressing the point 0)

$$\det[\langle N, \nabla_{v_i} v_j \rangle] \quad (4.3)$$

and our goal is, as it were in the previous section, to express (4.3) with the help of the metric rather than N ; we want to get rid of N . We therefore study the operator ∇_X and it is useful to decompose it into a tangential and a normal component.

4.1 Directional derivatives

See [1], Section 4.1.

4.2 The Lie bracket

See [1], Section 4.2.

4.3 The covariant derivative

See [1], Section 4.3.

Part V. The Riemannian curvature tensor

5 The Riemannian curvature tensor

5.1 Some algebra

See [1], Section 5.1.

5.2 The Riemannian curvature tensor

See [1], Section 5.2.

Recall.

- For coordinate fields X_i we write

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k,$$

where Γ_{ij}^k are the *Christoffel symbols*.

- First fundamental form (metric components):

$$g_{ij} = \langle X_i, X_j \rangle.$$

- $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$.

Proposition 4.3.5.

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell} g^{k\ell} \left(\partial_{x_i} g_{j\ell} + \partial_{x_j} g_{i\ell} - \partial_{x_\ell} g_{ij} \right).$$

Key point I: Γ_{ij}^k may be computed in terms of partial derivatives of the metric.

Riemann curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Computation:

$$R(X, Y)Z = \underbrace{\langle \partial_X Z, N \rangle \partial_Y N - \langle \partial_Y Z, N \rangle \partial_X N}_{\text{relates to the change of the normal vector}}.$$

Proposition 5.2.1. *The map*

$$(X, Y, Z) \mapsto R(X, Y)Z$$

is multilinear over $C^\infty(M)$.

Proof. Previous lecture. □

Consequence. The map is *pointwise*, i.e., a *tensor*.

Proposition 5.2.2. *We have*

$$\langle R(X, Y)Z, W \rangle = -\langle Z, R(X, Y)W \rangle.$$

Proof. (i)

$$\begin{aligned}\partial_X \partial_Y \langle Z, W \rangle &= \partial_X (\langle \nabla_Y Z, W \rangle + \langle Z, \nabla_Y W \rangle) \\ &= \langle \nabla_X \nabla_Y Z, W \rangle + \langle \nabla_Y Z, \nabla_X W \rangle \\ &\quad + \langle \nabla_X Z, \nabla_Y W \rangle + \langle Z, \nabla_X \nabla_Y W \rangle.\end{aligned}$$

(ii)

$$\begin{aligned}\partial_Y \partial_X \langle Z, W \rangle &= \langle \nabla_Y \nabla_X Z, W \rangle + \langle \nabla_X Z, \nabla_Y W \rangle \\ &\quad + \langle \nabla_Y Z, \nabla_X W \rangle + \langle Z, \nabla_Y \nabla_X W \rangle.\end{aligned}$$

So, subtracting (ii) from (i), the boxed terms cancel and

$$\begin{aligned}\partial_{[X,Y]} \langle Z, W \rangle &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W \rangle \\ &\quad + \langle Z, \nabla_X \nabla_Y W - \nabla_Y \nabla_X W \rangle.\end{aligned}$$

Also,

$$\partial_{[X,Y]} \langle Z, W \rangle = \langle \nabla_{[X,Y]} Z, W \rangle + \langle Z, \nabla_{[X,Y]} W \rangle.$$

So

$$\begin{aligned}\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle \\ = -\langle Z, \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W \rangle.\end{aligned}$$

□

Proposition 5.2.3. Let $M \subset \mathbb{R}^3$ be a manifold. For each point $p \in M$ the map

$$R_p: T_p M \times T_p M \longrightarrow \mathfrak{so}(T_p M)$$

defined by

$$[R(X, Y)Z]_p = R_p(X_p, Y_p) Z_p$$

is well defined.

($A \in \mathfrak{so}(V)$ means $\underbrace{\langle Au, v \rangle = -\langle u, Av \rangle}_{\text{skew symmetric}}.$)

Proof. R is tensorial, and use the previous proposition. □

Write

$$R(X_i, X_j)X_k = \sum_{\ell} R_{ijk}^{\ell} X_{\ell}.$$

Proposition 5.2.4.

$$R_{ijk}^{\ell} = \partial_{x_i} \Gamma_{jk}^{\ell} - \partial_{x_j} \Gamma_{ik}^{\ell} + \sum_m \left(\Gamma_{jk}^m \Gamma_{im}^{\ell} - \Gamma_{ik}^m \Gamma_{jm}^{\ell} \right).$$

Proof. Calculation, see book. □

Recall.

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell} g^{k\ell} \left(\partial_{x_i} g_{j\ell} + \partial_{x_j} g_{i\ell} - \partial_{x_\ell} g_{ij} \right).$$

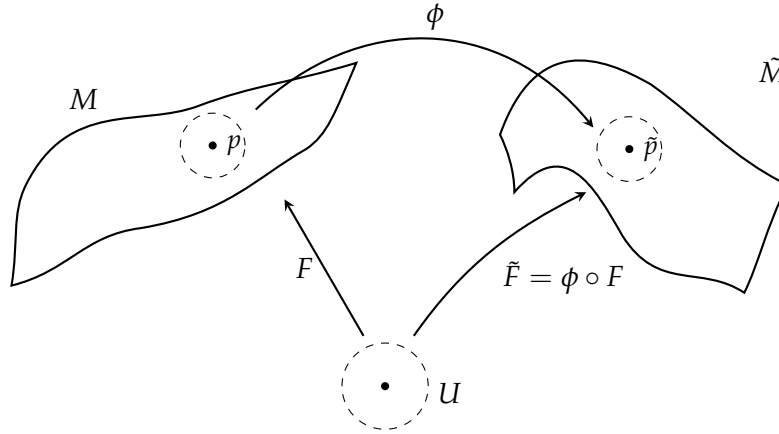
Key point II: R_{ijk}^ℓ may be computed in terms of partial derivatives of the metric.

Key point III: The metric is invariant under isometries (by definition).

Theorem 5.2.1. Let $\phi: M \rightarrow \tilde{M}$ be a local isometry between manifolds with curvature tensors R and $\tilde{R}$. Let $p \in M$, $\tilde{p} = \phi(p)$ and $A = d\phi_p$. Then for all $v_1, v_2, v_3 \in T_p M$

$$A(R_p(v_1, v_2) v_3) = \tilde{R}_{\tilde{p}}(Av_1, Av_2) Av_3. \quad (*)$$

Proof. Let $F: U \rightarrow M$ be a parametrisation and set $\tilde{F} = \phi \circ F$.



Representing R_p and $\tilde{R}_{\tilde{p}}$ with respect to X_1, X_2 and $\tilde{X}_1, \tilde{X}_2$ we have

$$R_p(X_i, X_j) X_k = \sum_{\ell} R_{ijk}^{\ell} X_{\ell},$$

$$\tilde{R}_{\tilde{p}}(\tilde{X}_i, \tilde{X}_j) \tilde{X}_k = \sum_{\ell} \tilde{R}_{ijk}^{\ell} \tilde{X}_{\ell},$$

$$\tilde{R}_{\tilde{p}}(AX_i, AX_j) AX_k = \sum_{\ell} \tilde{R}_{ijk}^{\ell} AX_{\ell} = A \left(\sum_{\ell} \tilde{R}_{ijk}^{\ell} X_{\ell} \right).$$

So the result follows if $\tilde{R}_{ijk}^{\ell} = R_{ijk}^{\ell}$.

Because of Key point II it is now enough to compare the metrics given by the two parametrisations:

$$g_{ij} = \langle X_i, X_j \rangle \stackrel{\text{isometry}}{=} \langle AX_i, AX_j \rangle = \langle \tilde{X}_i, \tilde{X}_j \rangle = \tilde{g}_{ij}.$$

Since $(*)$ now holds for all basis vectors X_i, X_j, X_k , it holds for all v_1, v_2, v_3 by multilinearity. $\square$

5.3 Theorema Egregium

See [1], Section 5.3.

For vector fields X, Y on M we define $\alpha(X, Y)$ by

$$\partial_X Y = \nabla_X Y + \underbrace{\alpha(X, Y)}_{\text{normal field}}.$$

Proposition 5.3.1.

$$\alpha(X, Y) = \alpha(Y, X).$$

Proof. Follows since we know that $[X, Y]$ is tangent:

$$[X, Y] = \partial_X Y - \partial_Y X = \nabla_X Y - \nabla_Y X + \alpha(X, Y) - \alpha(Y, X). \quad \square$$

Proposition 5.3.2. *The map*

$$(X, Y) \longmapsto \alpha(X, Y)$$

is bilinear over $C^\infty(M)$, i.e., a tensor.

Proof.

- $\alpha(fX, Y) = (\partial_{fX} Y)^{\text{norm}} = (f \partial_X Y)^{\text{norm}} = f (\partial_X Y)^{\text{norm}} = f \alpha(X, Y).$
- By symmetry, $\alpha(X, fY) = \alpha(fY, X) = f \alpha(Y, X) = f \alpha(X, Y).$ $\square$

Similarly to before:

$$\alpha_p: T_p M \times T_p M \longrightarrow (T_p M)^\perp$$

is well defined.

Recall that for a regular surface $M \subset \mathbb{R}^3$ and $p \in M$ we have defined

$$\underbrace{\Pi_p: T_p M \times T_p M \longrightarrow \mathbb{R}}_{\text{second fundamental form}}, \quad \Pi_p(u, v) = \langle \underbrace{W_p}_{\text{Weingarten map}} u, v \rangle.$$

Proposition 5.3.4. *Let $M \subset \mathbb{R}^3$ be a regular surface and let $N: M \rightarrow \mathbb{R}^3$ be a smooth unit normal near $p \in M$. For $u, v \in T_p M$ we have*

$$\alpha_p(u, v) = \Pi_p(u, v) \cdot N_p.$$

Proof.

- $X_p = u, Y_p = v.$
- $0 = \partial_X \langle Y, N \rangle = \langle \partial_X Y, N \rangle + \langle Y, \partial_X N \rangle.$

We get

$$\begin{aligned} \langle \alpha_p(u, v), N_p \rangle &= \langle \partial_X Y, N \rangle_p = -\langle Y, \partial_X N \rangle_p \\ &= -\langle v, dN_p(u) \rangle \\ &= \langle v, W_p(u) \rangle = \Pi_p(u, v). \end{aligned} \quad \square$$

Theorem 5.3.1 (Gauss equation). *If X, Y, Z, W are smooth vector fields on M , then*

$$\langle R(X, Y)Z, W \rangle = \langle \alpha(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle. \quad (**)$$

Proof. See book. $\square$

Theorem 5.3.2. *For v_1, v_2 orthonormal at p we have*

$$K(p) = -\langle R_p(v_1, v_2) v_1, v_2 \rangle.$$

Proof. Let $[a_{ij}]$ be the matrix of $W_p: T_pM \rightarrow T_pM$ with respect to v_1, v_2 . Then

$$a_{ij} = \langle v_i, W_p(v_j) \rangle = \Pi(v_i, v_j).$$

We have

$$\alpha_p(v_i, v_j) = a_{ij} \cdot N_p.$$

From (**) we get

$$-\langle R_p(v_1, v_2) v_1, v_2 \rangle = a_{11}a_{22} - a_{12}^2 = \det(W_p) = K(p). \quad \square$$

Theorem 5.3.3 (Theorema Egregium). *Let $\phi: M \rightarrow \tilde{M}$ be a local isometry between surfaces in $\mathbb{R}^3$, and let $p \in M$, $\tilde{p} = \phi(p)$. Then*

$$K(p) = \tilde{K}(\phi(p)).$$

Proof. Set $A = d\phi_p$, and let v_1, v_2 be an orthonormal basis at p . Then

$$\begin{aligned} K(p) &= -\langle R_p(v_1, v_2) v_1, v_2 \rangle = -\langle A(R_p(v_1, v_2) v_1), Av_2 \rangle \\ &= -\langle \tilde{R}_{\tilde{p}}(Av_1, Av_2) Av_1, Av_2 \rangle = \tilde{K}(\tilde{p}). \end{aligned} \quad \square$$

Part VI. Topics still to come

6 Geodesics

7 Differential forms

8 The Gauss-Bonnet Theorem

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References

- [1] K. Frøyshov, *Introduction to Differential Geometry*, UiO, 2026.