

MAT4510 - FALL 2026 - COURSE NOTES PART I - PLANE GEOMETRY

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1. INTRODUCTION

The following are course *notes* for MAT4510, Fall, 2026. The word "notes" reflects that they are partly notes intended for the lecturer himself, and partly a supplement for the student to attending classes and reading other literature.

2. AXIOMS FOR PLANE GEOMETRY

The presentation of Euclid's and Hilbert's axioms for plane geometry follows very closely and often verbatim the book [3], Chapter I and III. Another reference is [4] Chapter I.

The point will be to state and compare to some extent the two sets of axioms, not to go into details about set theory and logic. We will also consider some models for different constellations of axioms.

2.1. Undefined terms.

point
line
lies on (is contained in, is a point in)
between
congruent

2.2. Euclid's postulates/axioms as presented by Greenberg [3].

Definition 2.1. (Segment) Given two points a and b and a line l incident with a, b . The *segment* ab consists of the points a and b and all points c incident with l such that c is between a and b . The two given points a and b are called the *endpoints* of the segment ab .

Euclid I For every point p and for every point q not equal to p there exists a unique line l , denoted by $\overleftrightarrow{pq}$, that passes through p and q .

Euclid II For every segment ab and for every segment cd there exists a unique point e such that b is between a and e and segment cd is congruent to segment be .

Definition 2.2. (Circle) Given two points o and a . The set of all points p such that segment op is congruent to segment oa is called a circle with o as center, and each of the segments op is called a radius of the circle.

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Euclid III For every point o and every point a not equal to o there exists a circle with center o and radius oa .

Definition 2.3. (Ray) The ray $\overrightarrow{ab}$ is the following set of points lying on the line $\leftrightarrow ab$: those points that belong to the segment ab and all points c on ab such that b is between a and c . The ray $\overrightarrow{ab}$ is said to emanate from the vertex a and to be part of line $\leftrightarrow ab$.

Definition 2.4. (Opposite rays) Rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are opposite if they are distinct, if they emanate from the same point a , and if they are part of the same line $\leftrightarrow ab = \leftrightarrow ac$.

Definition 2.5. (Angle) An *angle with vertex a* is a point a with two distinct nonopposite rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$. The nonopposite rays are called the *sides* of the angle. We use notations $\sphericalangle a$, $\sphericalangle bac$ and $\sphericalangle cab$ for the angle.

Definition 2.6. (Supplementary angle) If two angles $\sphericalangle bad$ and $\sphericalangle cad$ have a common side $\overrightarrow{ad}$ and the other two sides $\overrightarrow{ab}$ and $\overrightarrow{ac}$ form opposite rays, the angles are *supplements of each other*, or *supplementary angles*.

Definition 2.7. (Right angle) An angle $\sphericalangle abc$ is a *right angle* if it has a supplementary angle to which it is congruent.

Euclid IV All right angles are congruent to each other.

Definition 2.8. (Parallel lines) Two lines l and m are *parallel* if they do not intersect, *i.e.*, if no point lies on both of them.

The following is a version of Euclid's postulate V called *Playfair's postulate*.

Parallel postulate For every line l and for every point p that does not lie on l there exists a unique line m through p that is parallel to l .

The following is Euclid's version of the parallel postulate presented in Greenberg, [3], page 128. It requires some definitions we haven't given.

Euclid V If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180° , then the two lines meet on that side of the transversal.

Hilbert's axiom of parallelism For every line l and every point P not lying on l there is at most one line m through p such that m is parallel to l .

See Greenberg, [3], page 102. In Greenberg, Corollary 2 to Theorem 4.1, Chapter 4, page 117, the existence of at least one parallel line is given (using other postulates of Hilbert).

2.3. Some configuration sets.

We consider now some sets with possible concrete notions of lines, between and congruent. We will use notation from set theory. So $B \subset A$ will mean that B is a subset of A , *i.e.*, a certain collection of points in A , and $a \in A$ will signify that a lies on A - is an element of A - and is also a point in B . The notation $a * c * b$ will signify that c lies between a and b .

The real line $\mathbb{R}$

A set $l \subset \mathbb{R}$ is a line if and only if $l = \mathbb{R}$. We say that a point $c \in \mathbb{R}$ is between two points a and b if $a \leq c \leq b$. We say that two segments (resp. angles) $A, B \subset \mathbb{R}$ are congruent if there exists a real number c such that, setting

$$f(x) = z + c \quad (2.1)$$

we have that $f(A) = B$. Note that the set of transformations on the form (2.1) constitutes a group.

The circle S

Let

$$S = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$$

The only line in S is S itself. For any three points $a, b, c \in S$ we declare that $a * c * b$. We say that two segments (resp. angles) $A, B \subset S$ are congruent if there exists a real number θ such that, setting

$$f(z) = e^{i\theta} z \quad (2.2)$$

we have that $f(A) = B$. Note that the set of transformations on the form (2.2) constitutes a group.

The Euclidean Plane $\mathbb{C}$

A *line* in $\mathbb{C}$ is a set

$$l = \{z \in \mathbb{C} : \operatorname{Re}(az + b) = 0, a, b \in \mathbb{C}, a \neq 0\}$$

We say that $z_0 * z_1 * z_2$ if

$$\operatorname{dist}(z_0, z_2) = \operatorname{dist}(z_0, z_1) + \operatorname{dist}(z_1, z_2)$$

We say that two segments (resp. angles) in $\mathbb{C}$ are *congruent* if there exists a transformation (called a *direct transformation* by Greenberg)

$$f(z) = e^{i\theta} z + b, \theta \in \mathbb{R}, b \in \mathbb{C} \quad (2.3)$$

with $f(A) = B$. Note that the set of transformations on the form (2.3) constitutes a group.

The sphere S^2

We define

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

A line in S^2 is a set

$$l = \{a = (x, y, z) \in S^2 : \alpha x + \beta y + \gamma z = 0 \text{ with } \alpha, \beta, \gamma \in \mathbb{R} \text{ not all zero}\}$$

For any $a, b, c \in l$ we say that $a * c * b$. We say that two segments (resp. angles) in S^2 are *congruent* if there exists $O \in SO(3)$ such that $O(A) = B$ ¹.

The projective plane P^2

An element of P^2 is a set $\{a, b\} \subset S^2$ with $b = -a$, i.e., we have that P^2 is the collection of all such sets². There is then a natural *projection*

$$\pi : S^2 \rightarrow P^2 \text{ where } \pi(a) = \{a, -a\}$$

If $a \sim b, a, b \in S^2$ we have that $O(a) \sim O(b)$ for any $O \in SO(3)$, so any such O *descends* as a transformation $\hat{O}$ on P^2 , i.e., there is a (unique) transformation $\hat{O} : P^2 \rightarrow P^2$ such that $\pi \circ O = \hat{O} \circ \pi$.

A line $\hat{l}$ is a set $\hat{l} = \pi(l)$ where l is a line in S^2 . For any three points $a, b, c \in \hat{l}$ we say that $a * c * b$. We say that two segments (resp. angles) in P^2 are *congruent* if there exists $\hat{O} \in \widehat{SO}(3)$ such that $\hat{O}(A) = B$.

The Poincaré disk Δ

We define

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

For a point $re^{i\theta} \in \Delta, r \neq 0$, let $S_{r,\theta} \subset \mathbb{C}$ denote the circle with a diameter being the segment between $re^{i\theta}$ and $r^{-1}e^{i\theta}$. Let $l_{r,\theta} = S_{r,\theta} \cap \Delta$. When $r \rightarrow 0$ we have that $l_{r,\theta}$ "approaches" the straight line segment passing through the origin with an angle $\theta + \frac{\pi}{2}$ to the real axis. Let $l_{0,\theta}$ denote this line segment. The lines l in Δ are the sets $l_{r,\theta}, 0 \leq r < 1, \theta \in \mathbb{R}$. They are circles (or lines) in $\mathbb{C}$ intersecting the boundary $b\Delta$ at right angles at exactly two points. For a line $l_{r,\theta}$ with $r = 0$, betweenness is defined as it is on a line in $\mathbb{C}$, and for $r > 0$ we say that $a * c * b$ if c lies between a and b in a clockwise sense.

For the notion of congruence in Δ we need a certain group of transformations Δ , just like we did in the cases of $\mathbb{R}, S^1, \mathbb{C}, S^2$ and P^2 . For $a \in \Delta$ with and $\theta \in \mathbb{R}$, we set

$$\phi(z) = e^{i\theta} \frac{z + a}{1 + \bar{a}z} \text{ and } \psi(z) = e^{-i\theta} \frac{z - ae^{i\theta}}{1 - \overline{ae^{i\theta}}z} \quad (2.4)$$

The maps (2.4) are holomorphic transformation of Δ , and $\psi \circ \phi(z) = \phi \circ \psi(z) = z$. Thus the set of transformations on the form of ϕ constitutes a group of transformations of Δ , denoted by $\text{Aut}_{\text{hol}}\Delta$. A key feature of this group is that the transformations ϕ send lines to lines (see Section 4 for more details about this group). Finally, we say that two segments (resp. angles) $A, B \subset \Delta$ are congruent if there exists $\phi \in \text{Aut}_{\text{hol}}\Delta$ with $\phi(A) = B$.

¹We have that $SO(3)$ denotes the group of orthogonal 3×3 -matrices O such that $\det(O) = 1$. We identify the set of matrices with the corresponding set of transformations of $\mathbb{R}^3$. Such a matrix leaves the sphere S^2 invariant.

²In set theory there is no difference between the sets $\{a, b\}$ and $\{b, a\}$; two sets A and B are equal if and only if for all x we have that $x \in A$ if and only if $x \in B$.

3. COMPLETION OF EUCLID'S AXIOMS - HILBERT'S AXIOMS

3.1. Axioms of incidence. (See Chapter 2, pp. 50-51, Greenberg [3].)

Incidence axiom I For every point p and for every point q not equal to p there exists a unique line l incident with p and q .

Incidence axiom II For every line l there exist at least two distinct point incident with l .

Incidence axiom III There exist three distinct points with the property that no line is incident with all three of them.

3.2. Axioms of betweenness.

Betweenness axiom I If $a * b * c$, then a, b, c are three distinct points lying on the same line, and $c * b * a$.

Betweenness axiom II Given two distinct points b, d , there exist points a, c, e lying on $\overleftrightarrow{bd}$ such that

$$a * b * d \text{ and } b * c * d \text{ and } c * d * e$$

Betweenness axiom III If a, b, c are three distinct points lying on the same line, then one and only one of the points is between the other two.

Definition 3.1. (Plane separation) Let l be a line and let a, b be points not on l . If $a = b$ or if ab does not intersect l , we say that a and b are on the *same side* of l . If $a \neq b$ and ab intersects l we say that a and b are on *opposite sides* of l .

Remark 3.2. If a and b are points then, by the law of the excluded middle, we have that $a = b$ or $a \neq b$ (this does not hold in so called *intuitionistic* or *constructive* mathematics). Further, if $a \neq b$ we have that ab intersects l or it does not. So either a and b are on the same side of l or they are on opposite sides.

Betweenness axiom IV For any line l and any three points a, b, c not lying on l we have:

- (i) If a and b are on the same side of l and if b and c are on the same side of l , then a and c are on the same side of l .
- (ii) If a and b are on opposite sides of l , and b and c are on opposite sides of l , then a and c are on the same side of l .

Definition 3.3. Let $\angle cab$ be an angle and let d be a point. We say that d is in the *interior* of $\angle cab$ if d is on the same side of $\overleftrightarrow{ac}$ as b and if d is on the same side of $\overleftrightarrow{ab}$ as c .

Remark 3.4. (Warning, Greenberg, [3], page 81.) As an example of statement that might be tempting to assume that holds consider the following: *every point in the interior of an angle lies on a segment joining a point on one side of the angle (what does that mean?) to a point on the other side (what does that mean?).*

3.3. Axioms of congruence.

Definition 3.5. (Greenberg, [3], page 83.) Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are and corresponding angles are congruent. When we write $\Delta abc \equiv \Delta def$ we understand that a corresponds to d , b to e , and c to f .

Congruence axiom I If a and b are distinct points, and if a' is any point, then for each ray r emanating from a' there is a unique point b' on r such that $b' \neq a'$ and $ab \equiv a'b'$.

Congruence axiom II If $ab \equiv cd$ and $ab \equiv ef$, then $cd \equiv ef$. Moreover, every segment is congruent to itself.

Congruence axiom III If $a * b * c$, $a' * b' * c'$, $ab \equiv a'b'$, and $bc \equiv b'c'$, then $ac \equiv a'c'$.

Congruence axiom IV Given any $\angle bac$, and given a ray $\overrightarrow{a'b'}$ emanating from a point a' , then there is a *unique* ray $\overrightarrow{a'c'}$ on a given side of the line $\overleftrightarrow{a'b'}$ such that $\angle b'a'c' \equiv \angle bac$.

Congruence axiom V If $\angle a \equiv \angle b$ and $\angle a \equiv \angle c$ then $\angle b \equiv \angle c$. Moreover any angle is congruent to itself.

Congruence axiom VI (SAS) If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

3.4. Axioms of continuity.

Continuity axiom I (Circular continuity principle, [3], page 94.) If a circle γ has one point inside and one point outside another circle γ' , then the two circles intersect in two points.

Continuity axiom II (Dedekind's axiom, [3], page 98.) Suppose that the set $\{l\}$ of all points on a line l is the disjoint union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets, such that no point of either subset is between two points of the other. Then there exists a unique point o on l such that one of the subsets is equal to a ray of l with vertex o and the other subset is equal to the complement.

Axiom of Archimedes ([3], page 95.) If cd is any segment, a any point, and r any ray with vertex a , then for every point $b \neq a$ on r there is a number n such that when cd is laid off n times on r starting at a , a point e is reached such that $n \cdot cd \equiv ae$ and either $b = e$ or $a * b * e$.

3.5. Axiom of parallelism.

Axiom of parallelism ([3], page 102.) For every line l and every point p not lying on l there is at most one line m through p such that m is parallel to l .

4. THE HYPERBOLIC PLANE

As a set, the *hyperbolic plane* is the Euclidean unit disk $\Delta \subset \mathbb{C}$.

4.1. Lines in the hyperbolic plane.

Definition 4.1. (Hyperbolic line) Let $re^{i\theta} \in \Delta, r \neq 0, \theta \in \mathbb{R}$. We set

$$l_{r,\theta} = \{z \in \Delta : |z - \frac{1}{2}(r + r^{-1})e^{i\theta}| = \frac{1}{2}(r^{-1} - r)\} \quad (4.1)$$

For $r = 0$ we define

$$l_{r,\theta} = l_{0,\theta} = \{se^{i\theta + \frac{\pi}{2}} : -1 < s < 1\}$$

A *line* in Δ is a set $l_{r,\theta}$ (note that $re^{i\theta} \in l_{r,\theta}$).

Remark 4.2. A line $l_{r,\theta}$ is an arc in a Euclidean circle $S_{r,\theta}$ (see (4.1)) and this circle intersects the boundary of Δ at two right angles. For $r = 0$ the "circle" is a Euclidean line. The only lines that pass through the origin are the Euclidean "lines" $l_{0,\theta}$.

We set

$$l_{\mathbb{R}} = l_{0,\frac{\pi}{2}} = \Delta \cap \mathbb{R}$$

If for $a \in \Delta$ we write $a > 0$ or $a < 0$ we implicitly mean that $a \in l_{\mathbb{R}}$.

4.2. The holomorphic automorphism group of the unit disk.

For $a \in \Delta$ and $\theta \in \mathbb{R}$ we define

$$\phi(z) = \phi_{a,\theta} = e^{i\theta} \cdot \frac{z - a}{1 - \bar{a}z} \quad (4.2)$$

If $\theta = 0$ we denote this map by ϕ_a , and by writing ϕ_a it is for us always assumed that $a \in \Delta$.

Theorem 4.3. *The set $\text{Aut}_{\text{hol}}\Delta$ of maps of the form (4.2) forms a group of conformal³ automorphisms of Δ under composition.*

Definition 4.4. Let $a, b, c \in \Delta$ be distinct points. We say that $a * b * c$ if and only if $\phi_b(b) = 0$ is between $\phi_b(a)$ and $\phi_b(c)$ in the Euclidean sense.

Definition 4.5. (Congruence of line segments, see Definition 2.1) We say that line segments ab and cd in Δ are *congruent* if there exists $\phi \in \text{Aut}_{\text{hol}}\Delta$ such that $\phi(a) = c$ and $\phi(b) = d$.

Definition 4.6. (Congruence of angles, see Definition 2.5) We say that two angles $\angle bac$ and $\angle edf$ are *congruent* if there exists $\phi \in \text{Aut}_{\text{hol}}\Delta$ such that $\phi(\angle bac) = \angle edf$.

For the proof of Theorem 4.3 we need the following

³A map is conformal if its differential at any point preserves angles. A complex analytic function is conformal provided its differential is never zero.

Lemma 4.7. *A map (4.2) satisfies $\phi(\bar{\Delta}) \subset \bar{\Delta}$.*

Proof. We have that

$$\begin{aligned}
1 - |\phi_a(z)|^2 &= 1 - \frac{(z-a)(\bar{z}-\bar{a})}{(1-\bar{a}z)(1-a\bar{z})} = 1 - \frac{|z|^2 - 2\operatorname{Re}(\bar{a}z) + |a|^2}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} \\
&= 1 - \frac{|z|^2 - 1 + 1 - 2\operatorname{Re}(\bar{a}z) + |a|^2}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} \\
&= \frac{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2 - (|z|^2 - 1 + 1 - 2\operatorname{Re}(\bar{a}z) + |a|^2)}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} \\
&= \frac{|a|^2|z|^2 - |z|^2 + 1 - |a|^2}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} = \frac{1 - |a|^2 - |z|^2(1 - |a|^2)}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} \\
&= \frac{(1 - |z|^2)(1 - |a|^2)}{1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2} \geq 0 \text{ for } |z| < 1
\end{aligned}$$

where the last inequality follows since

$$1 - 2\operatorname{Re}(\bar{a}z) + |a|^2|z|^2 \geq 1 - 2|\operatorname{Re}(\bar{a}z)| + |a|^2|z|^2 \geq 1 - 2|a||z| + |a|^2|z|^2 = (1 - |a||z|)^2 \geq 0$$

□

Proof of Theorem 4.3. By Lemma 4.7 we have that $\phi(\Delta) \subset \Delta$. Straight forward computations shows that the map

$$\psi(z) = e^{-i\theta} \frac{z - ae^{i\theta}}{1 - \overline{ae^{i\theta}}z}$$

satisfies $\psi(\phi(z)) = \phi(\psi(z)) = z$, and that a composition of two maps of the general form (4.2) is again of the same form. So the set $\operatorname{Aut}_{\text{hol}}\Delta$ of maps of the form (4.2) forms a group under composition. □

Remark 4.8. It can be shown that any bijective holomorphic map $\phi : \Delta \rightarrow \Delta$ is on the form (4.2). Therefore what we have defined as $\operatorname{Aut}_{\text{hol}}\Delta$ is in fact *the* group of holomorphic automorphisms of Δ . Note that the maps on the form (4.2) are in fact automorphisms of the closed disk $\bar{\Delta}$.

Lemma 4.9. *Let $a, b \in \Delta$ with $a \neq b$. Then there is a unique $\psi \in \operatorname{Aut}_{\text{hol}}\Delta$ with*

$$\psi(a) = 0 \text{ and } \psi(b) > 0 \text{ and we have } \psi(b) = \left| \frac{b-a}{1-\bar{a}b} \right|$$

Proof. Start by setting $\phi(z) = \frac{z-a}{1-\bar{a}z}$. We then have that $\phi(a) = 0$ and we set $re^{i\theta} = \phi(b)$. Then $\psi(z) = e^{-i\theta} \cdot \phi(z)$ satisfies $\psi(a) = 0$ and $\hat{b} = \psi(b) > 0$. We have that ψ is unique since any automorphism ϕ satisfying $\phi(0) = 0$ and $\phi(\hat{b}) > 0$ is the identity map. □

Definition 4.10. Let $a, b \in \Delta$. We define the *Möbius distance*

$$m(a, b) = \left| \frac{b-a}{1-\bar{a}b} \right|$$

Remark 4.11. Note that $m(a, b) \geq d(a, b)$ (Euclidean distance) and that we have equality if and only if a or b is zero.

Lemma 4.12. Let $a, b \in \Delta$ and let $\phi \in \text{Aut}_{\text{hol}}\Delta$. Then $m(a, b) = m(\phi(a), \phi(b)) =: m(a'b')$.

Proof. Let ψ be the unique map from Lemma 4.9. Then $\psi \circ \phi^{-1}$ is the unique map corresponding to a', b' . $\square$

Lemma 4.13. We have that m is a distance.

Proof. Clearly $m(a, b) \geq 0$ with equality if and only if $a = b$. We have that

$$m(a, b) = \frac{|b - a|}{|1 - \bar{a}b|} = \frac{|a - b|}{|1 - \bar{a}b|} = \frac{|a - b|}{|1 - \bar{b}a|} = m(b, a)$$

For the triangle inequality let $a, b, c \in \Delta$, let ψ be the map from Lemma 4.9, and set $b' = \psi(b)$ and $c' = \psi(c)$. Recall that m is ψ -invariant. By the Euclidean triangle inequality and that m dominates d we have that

$$m(a, b) = d(0, b') \leq d(0, c') + d(c', b') = m(0, c') + d(c', b') \leq m(0, c') + m(c', b')$$

$\square$

Remark 4.14. Unlike in the Euclidean case we cannot define betweenness directly using this distance, *i.e.*, we cannot say that $a * b * c$ provided that

$$m(a, c) = m(a, b) + m(b, c)$$

In fact, for three distinct points $a, b, c \in \Delta$ we always have that

$$m(a, c) < m(a, b) + m(b, c)$$

Lemma 4.15. Let $\phi \in \text{Aut}_{\text{hol}}\Delta$ satisfy $\phi(-1) = -1$ and $\phi(1) = 1$. Then

$$\phi(z) = \frac{z + r}{1 + rz} \text{ with } r \text{ real}$$

Proof. We have

$$\phi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \text{ for } a \in \Delta$$

Then

$$\begin{aligned} 0 &= e^{-i\theta}(\phi(-1) + \phi(1)) = \frac{-1 - a}{1 + \bar{a}} + \frac{1 - a}{1 - \bar{a}} = \frac{(1 - a)(1 + \bar{a}) - (1 + a)(1 - \bar{a})}{(1 + \bar{a})(1 - \bar{a})} \\ &= \frac{1 - a + \bar{a} - a\bar{a} - (1 + a - \bar{a} - a\bar{a})}{(1 + \bar{a})(1 - \bar{a})} = \frac{-2(a - \bar{a})}{1 - a\bar{a}} \end{aligned}$$

and so a is a real number. Then $\theta = 0$. $\square$

Lemma 4.16. *The map $\phi(z) = \frac{z+r}{1+rz}$, $r \in (-1, 1)$, leaves $l_{\mathbb{R}}$ invariant, and for $-1 < s < t < 1$*

we have that

$$-1 = \phi(-1) < \phi(s) < \phi(t) < \phi(1) = 1$$

Proof. A simple computation shows that $\phi(-1) = -1$ and $\phi(1) = 1$. We have that

$$\begin{aligned} \phi(t) - \phi(s) &= \frac{t+r}{1+rt} - \frac{s+r}{1+sr} = \frac{(t+r)(1+sr) - (s+r)(1+rt)}{(1+rt)(1+sr)} \\ &= \frac{t+tsr+r+sr^2 - (s+srt+r+r^2t)}{(1+rt)(1+sr)} \\ &= \frac{t-s+(s-t)r^2}{(1+rt)(1+sr)} = \frac{(1-r^2)(t-s)}{(1+rt)(1+sr)} \end{aligned}$$

□

Lemma 4.17. *Fix $\theta \in \mathbb{R}$. Then the maps*

$$\phi_t(z) = \frac{z + te^{i\theta}}{1 + te^{-i\theta}z} \text{ with } t \in (-1, 1) \quad (4.3)$$

leave the line $l_{0, \theta + \frac{\pi}{2}}$ invariant, and ϕ_t maps the line $l_{0, \theta}$ to the line $l_{t, \theta}$.

Proof. First we have that

$$\phi_t(se^{i\theta}) = \frac{se^{i\theta} + te^{i\theta}}{1 + te^{-i\theta}se^{i\theta}} = \frac{(s+t)e^{i\theta}}{1+ts}$$

Next, the radius of the claimed circle is $\frac{1}{2}(t^{-1} - t)$ and its centre is

$$te^{i\theta} + \frac{1}{2}(t^{-1} - t)e^{i\theta} = \frac{1}{2}(t + t^{-1})e^{i\theta}$$

We have that

$$\phi_t(se^{i(\theta + \frac{\pi}{2})}) = \frac{se^{i(\theta + \frac{\pi}{2})} + te^{i\theta}}{1 + te^{-i\theta}se^{i(\theta + \frac{\pi}{2})}} = e^{i\theta} \frac{se^{i\frac{\pi}{2}} + t}{1 + tse^{i\frac{\pi}{2}}} = e^{i\theta} \frac{si + t}{1 + tsi}$$

and so

$$\begin{aligned} |\phi_t(se^{i(\theta + \frac{\pi}{2})}) - \frac{1}{2}(t + t^{-1})e^{i\theta}| &= \left| \frac{si + t}{1 + tsi} - \frac{1}{2}(t + t^{-1}) \right| = \frac{1}{2} \left| \frac{2si + 2t - (1 + tsi)(t + t^{-1})}{1 + tsi} \right| \\ &= \frac{1}{2} \frac{2si + 2t - t - t^{-1} - t^2si - si}{1 + tsi} = \frac{1}{2} \frac{si + t - t^{-1} - t^2si}{1 + tsi} \\ &= \frac{1}{2} \frac{si + t^{-1}(1 - t^2) - t^2si}{1 + tsi} = \frac{1}{2} \frac{si(1 - t^2) + t^{-1}(1 - t^2)}{1 + tsi} \\ &= \frac{1}{2} \frac{(si + t^{-1})(1 - t^2)}{1 + tsi} = \frac{1}{2} \frac{t^{-1}(tsi + 1)(1 - t^2)}{1 + tsi} \\ &= \frac{1}{2} t^{-1}(1 + t^2) \end{aligned}$$

□

Definition 4.18. For $t \neq 0$ we call the maps (4.3) *translations* (along the line $l_{0,\theta+\frac{\pi}{2}}$). The maps $z \mapsto e^{i\theta}z$ are called *rotations* at the origin in Δ .

Lemma 4.19. Assume $e_1, e_2 \in b\Delta$. Then there exists $\phi \in \text{Aut}_{\text{hol}}\Delta$ such that $\phi(-1) = e_1, \phi(1) = e_2$ and $\phi(l_{\mathbb{R}})$ is a line with end points e_1 and e_2 .

Proof. Consider

$$\phi(z) = \frac{z + ir}{1 - irz}$$

We have that

$$\phi(1) = \frac{1 - r^2}{1 + r^2} + i \frac{2r}{1 + r^2} \text{ and } \phi(-1) = \frac{r^2 - 1}{1 + r^2} + i \frac{2r}{1 + r^2}$$

This shows that for any $0 < \theta \leq \frac{\pi}{2}$ we may achieve that

$$\phi(-1) = e^{i(\frac{\pi}{2} + \theta)} \text{ and } \phi(1) = e^{i(\frac{\pi}{2} - \theta)}$$

Composing with a rotation we may achieve that $\phi(-1) = e_1$ and $\phi(1) = e_2$ or vice versa. In the latter case consider the map $z \mapsto \phi(-z)$. □

Lemma 4.20. Let $\psi \in \text{Aut}_{\text{hol}}\Delta$. Then $\psi(l_{\mathbb{R}})$ is a line passing through $e_1 = \psi(-1)$ and $e_2 = \psi(1)$.

Proof. Let ϕ be the map from Lemma 4.19, and set $f = \phi^{-1} \circ \psi$. Then

$$f(z) = \frac{z + r}{1 + rz}$$

by Lemma 4.15, and so $\psi(z) = \phi(f(z))$. □

Corollary 4.21. For $\phi \in \text{Aut}_{\text{hol}}\Delta$ and any line l we have that $\phi(l)$ is a line.

Corollary 4.22. Let $a, b \in \Delta$. Then there exist unique $r > 0$ and $\phi \in \text{Aut}_{\text{hol}}\Delta$ such that $\phi(0) = a$ and $\phi(r) = b$. We have that $l = \phi(l_{\mathbb{R}})$ is a line, and $r = m(a, b)$.

Proof. By Lemma 4.9 there exists a unique ψ with $\psi(a) = 0$ and $\psi(b) > 0$, and we have that $\phi(b) = m(a, b)$. Set $\phi = \psi^{-1}$. By Lemma 4.20 we have that $\phi(l_{\mathbb{R}})$ is a line. □

5. MANDATORY ASSIGNMENT

Problem 1 Find (wherever you want) a theorem of Euclid's whose proof does not strictly speaking follow from his axioms. Explain why.

For the following three problems it is (as allways) highly recommended to construct proofs without looking them up somewhere.

Problem 2 Prove the following using only Hilbert's axioms: Let l be a line. If a and b are on opposite sides of l and b and c are on the same side of l , then a and c are on opposite sides of l .

Suggested solution: We will use Betweenness axiom IV. If a and c are on the same side of l and if b and c are on the same side of l , then, by BAIV(i) we have that a and b are on the same side of l . Therefore, the assumptions imply that a and c cannot be on the same side of l . So a and c are on opposite sides of l .

Betweenness axiom IV For any line l and any three points a, b, c not lying on l we have:

- (i) If a and b are on the same side of l and if b and c are on the same side of l , then a and c are on the same side of l .
- (ii) If a and b are on opposite sides of l , and b and c are on opposite sides of l , then a and c are on the same side of l .

Definition 5.1. (Half-plane) Let l be a line and let a be a point not on l . We define $H_{a,l}$ to be the set of points lying on the same side of l as a .

Problem 3 Prove the following using only Hilbert's axioms: Every line bounds exactly two half-planes and these half-planes have no points in common.

Suggested solution:

- Let l be a line.
- Let $c \in l$.
- Let a be a point not on l .
- Let b be a point with $a * c * b$.
- We consider $H_{l,a}$ and $H_{l,b}$. Then b is not in $H_{l,a}$.
- Let d be a point not on l . If $d = a$ or if ad does not intersect l we have that $d \in H_{l,a}$.
- If d were also in $H_{l,b}$ we would have that $b \in H_{l,a}$ which it is not.
- Assume that d is on the opposite side of l . Then if d were in the opposite side of b we would have that b is in the same side as a , which it is not. So d is in $H_{l,b}$.

Problem 4 Prove the following using only Hilbert's axioms: Given $a * b * c$ and $a * c * d$ we have that $b * c * d$ and $a * b * d$.

Suggested solution:

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-
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Problem 5 Let $a, b, c \in \Delta$ and let $\phi \in \text{Aut}_{\text{hol}}\Delta$. Prove that

$$a * b * c \text{ if and only if } \phi(a) * \phi(b) * \phi(c)$$

Problem 6 Let a, b and c, d be two pairs of distinct points in Δ . Prove that $ab \equiv cd$ if and only if there exists $\phi \in \text{Aut}_{\text{hol}}\Delta$ such that $\phi(ab) = cd$.

Problem 7 Prove that Hilbert's Betweenness Axiom II holds in Δ .

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