Paracompactness

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Definition 1 *Let* $\mathbf{U} = \{U_\alpha\}_{\alpha \in I}$ *and* $\mathbf{V} = \{V_\beta\}_{\beta \in J}$ *be covers of a space* X*. Then* $\mathbf{V}$ *is called a* refinement *of* $\mathbf{U}$ *if there exists a map* $r : J \to I$ *such that*

$$V_\beta \subset U_{r(\beta)} \quad \textit{for all } \beta \in J.$$

Definition 2 *A collection* $\{U_\alpha\}_{\alpha \in I}$ *of subsets of a space* X *is called* locally finite *if each point in* X *has a neighbourhood* W *such that* $W \cap U_\alpha \neq \emptyset$ *for only finitely many* α*.*

Definition 3 *A topological space* X *is called* paracompact *if every open cover of* X *has an open, locally finite refinement.*

Note that some authors require paracompact spaces to be Hausdorff.

Proposition 1 *Every closed subspace of a paracompact space* X *is paracompact.*

Proof. Let $A \subset X$ be closed and $\{U_\alpha\}_{\alpha \in I}$ an open cover of A. Then $U_\alpha = A \cap U'_\alpha$ for some open subset $U'_\alpha \subset X$. Adding $X \setminus A$ to the collection $\{U'_\alpha\}_{\alpha \in I}$ yields a cover of X which has an open, locally finite refinement $\{V_\beta\}_{\beta \in J}$. Now, $\{A \cap V_\beta\}_{\beta \in J}$ is an open, locally finite refinement of the cover $\{U_\alpha\}_{\alpha \in I}$ of A. $\square$

Lemma 1 *If* $\{A_i\}_{i \in I}$ *is a locally finite collection of subsets of a space* X *then*

$$\overline{\bigcup_i A_i} = \bigcup_i \bar{A}_i.$$

Proof. We prove that the left-hand side is contained in the right-hand side, the reverse inclusion being obvious. Suppose

$$p \in X \setminus \bigcup_i \bar{A}_i = \bigcap_i (X \setminus \bar{A}_i).$$

Choose an (open) neighbourhood U of p and a finite subset $J \subset I$ such that $U \cap A_i = \emptyset$ for $i \in I \setminus J$. Then

$$V := \bigcap_{j \in J} \left(X \setminus \bar{A}_j\right)$$

is a neighbourhood of p which is disjoint from A_j for all $j \in J$. Hence, $U \cap V$ is a neighbourhood of p which is disjoint from A_i for all $i \in I$, so

$$p \notin \overline{\bigcup_i A_i}. \qquad \square$$

Proposition 2 *Every paracompact Hausdorff space X is normal.*

Proof. (i) We first show that X is regular. Let $A \subset X$ be closed and $q \in X \setminus A$. For each $p \in A$ choose disjoint neighbourhoods U_p and W_p of p and q, respectively. Adding $X \setminus A$ to the collection $\{U_p\}_{p \in A}$ gives an open cover of X which has an open, locally finite refinement $\{V_i\}_{i \in I}$. Set

$$J := \{i \in i \mid V_i \cap A \neq \emptyset\}, \tag{1}$$

$$W := \bigcup_{j \in J} V_j. \tag{2}$$

Then $A \subset W$. Since each V_j is contained in some U_p we have

$$q \notin \bigcup_{j \in J} \overline{V}_j = \overline{W}.$$

(ii) Let $A, B \subset X$ be disjoint closed sets. Arguing as in (i) with B in place of q shows that there exist disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$. $\square$

Definition 4 *A topological space is called* σ-compact *if it is a countable union of compact subsets.*

Theorem 1 *Let X be a topological space in which every point p has a neighbourhood U_p which is second countable and precompact (i.e. its closure is compact). Then among the following statements the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ hold:*

(i) *X is second countable.*

(ii) *X is σ-compact.*

(iii) *X is paracompact.*

(iv) *Every component of X is second countable.*

Proof. $(i) \Rightarrow (ii)$: Let X be second countable. Then every open cover of X has a countable subcover. Applying this to the open cover $\{U_p\}_{p \in X}$ we see that there is a sequence $p_1, p_2, \ldots$ in X such that

$$X = \bigcup_{k=1}^{\infty} U_{p_k}.$$

Since $\overline{U}_p$ is compact for each p it follows that X is σ-compact.

$(ii) \Rightarrow (iii)$: Let X be σ-compact, say $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. We first find a sequence

$$V_0 \subset V_1 \subset V_2 \subset \ldots$$

of precompact open subsets of X whose union is all of X, such that $\overline{V}_j \subset V_{j+1}$ for all j. Let $V_0 := \emptyset$. After $V_0, \ldots, V_j$ have been chosen, note that $K_j \cup \overline{V}_j$ is compact, so there are finitely many points $p_1, \ldots, p_m \in X$ such that

$$K_j \cup \overline{V}_j \subset U_{p_1} \cup \cdots \cup U_{p_m}.$$

Set $V_{j+1} := U_{p_1} \cup \cdots \cup U_{p_m}$.

Now let $\{W_i\}_{i \in I}$ be any open cover of X. Each set $\overline{V}_k \setminus V_{k-1}$ is compact and is therefore contained in $\bigcup_{i \in I_k} W_i$ for some finite subset $I_k \subset I$. Then

$$\{W_i \setminus \overline{V}_{k-2}\}_{k \in \mathbb{N}, i \in I_k}$$

is a locally finite, open refinement of $\{W_i\}_{i \in I}$.

$(iii) \Rightarrow (iv)$: Suppose X is paracompact and non-empty, and let Y be a component of X. We will show that Y is second countable. Since Y is closed in X, Proposition 1 says that Y is paracompact. Let $\{V_\alpha\}_{\alpha \in I}$ be a locally finite, open refinement of the open cover $\{U_p \cap Y\}_{p \in Y}$ of Y. Each $\overline{V}_\alpha$

is compact and can therefore be covered by finitely many $V_{\alpha'}$, $\alpha' \in I$. We can now find a sequence of finite subsets $J_0 \subset J_1 \subset J_2 \subset \cdots \subset I$ such that $J_0 = \{\alpha_0\}$ for some α_0 with V_{α_0} non-empty, and such that for each $k \geq 0$ one has

$$\bigcup_{\alpha \in J_k} \overline{V}_\alpha \subset \bigcup_{\beta \in J_{k+1}} V_\beta.$$

Set $J := \bigcup_k J_k$ and

$$Z := \bigcup_{\alpha \in J} V_\alpha.$$

Then $\overline{Z} = Z$ by Lemma 1, so Z is both open and closed. Because Z is non-empty and Y is connected we must have $Z = Y$. But then Y is a countable union of subspaces V_α each of which is second countable, hence Y is second countable. $\square$

Lemma 2 (Shrinking lemma) *Let X be a paracompact Hausdorff space and $\{U_i\}_{i \in I}$ an open cover of X. Then there exists an open cover $\{V_i\}_{i \in I}$ of X such that*

$$\overline{V}_i \subset U_i \quad \textit{for all } i \in I.$$

Proof. By Proposition 2 the space X is normal, so each $p \in U_i$ has a neighbourhood $W_{i,p}$ such that $\overline{W}_{i,p} \subset U_i$. The collection of all such sets $W_{i,p}$ with $i \in I$ and $p \in U_i$ is an open cover of X which has a locally finite, open refinement $\{Z_j\}_{j \in J}$. Choose a map $r : J \to I$ such that $\overline{Z}_j \subset U_{r(j)}$ for each $j \in J$, and set

$$V_i := \bigcup_{r(j)=i} Z_j.$$

Then by Lemma 1 we have

$$\overline{V}_i = \bigcup_{r(j)=i} \overline{Z}_j \subset U_i. \qquad \square$$