Paracompactness

Kim A. Frøyshov

**Definition 1** Let $U = \{ U_{\alpha} \}_{\alpha \in I}$ and $V = \{ V_{\beta} \}_{\beta \in J}$ be covers of a space $X$. Then $V$ is called a refinement of $U$ if there exists a map $r : J \to I$ such that

$$V_{\beta} \subset U_{r(\beta)} \quad \text{for all } \beta \in J.$$ 

**Definition 2** A collection $\{ U_{\alpha} \}_{\alpha \in I}$ of subsets of a space $X$ is called locally finite if each point in $X$ has a neighbourhood $W$ such that $W \cap U_{\alpha} \neq \emptyset$ for only finitely many $\alpha$.

**Definition 3** A topological space $X$ is called paracompact if every open cover of $X$ has an open, locally finite refinement.

Note that some authors require paracompact spaces to be Hausdorff.

**Proposition 1** Every closed subspace of a paracompact space $X$ is paracompact.

**Proof.** Let $A \subset X$ be closed and $\{ U_{\alpha} \}_{\alpha \in I}$ an open cover of $A$. Then $U_{\alpha} = A \cap U'_{\alpha}$ for some open subset $U'_{\alpha} \subset X$. Adding $X \setminus A$ to the collection $\{ U'_{\alpha} \}_{\alpha \in I}$ yields a cover of $X$ which has an open, locally finite refinement $\{ V_{\beta} \}_{\beta \in J}$. Now, $\{ A \cap V_{\beta} \}_{\beta \in J}$ is an open, locally finite refinement of the cover $\{ U_{\alpha} \}_{\alpha \in I}$ of $A$. \[\square\]

**Lemma 1** If $\{ A_{i} \}_{i \in I}$ is a locally finite collection of subsets of a space $X$ then

$$\bigcup_{i} A_{i} = \bigcup_{i} \bar{A}_{i}.$$ 

**Proof.** We prove that the left-hand side is contained in the right-hand side, the reverse inclusion being obvious. Suppose

$$p \in X \setminus \bigcup_{i} \bar{A}_{i} = \bigcap_{i} (X \setminus \bar{A}_{i}).$$

1
Choose an (open) neighbourhood $U$ of $p$ and a finite subset $J \subset I$ such that $U \cap A_i = \emptyset$ for $i \in I \setminus J$. Then

$$V := \bigcap_{j \in J} \left( X \setminus \overline{A_j} \right)$$

is a neighbourhood of $p$ which is disjoint from $A_j$ for all $j \in J$. Hence, $U \cap V$ is a neighbourhood of $p$ which is disjoint from $A_i$ for all $i \in I$, so

$$p \notin \bigcup_i A_i. \quad \square$$

**Proposition 2** Every paracompact Hausdorff space $X$ is normal.

**Proof.** (i) We first show that $X$ is regular. Let $A \subset X$ be closed and $q \in X \setminus A$. For each $p \in A$ choose disjoint neighbourhoods $U_p$ and $W_p$ of $p$ and $q$, respectively. Adding $X \setminus A$ to the collection $\{U_p\}_{p \in A}$ gives an open cover of $X$ which has an open, locally finite refinement $\{V_i\}_{i \in I}$. Set

$$J := \{ i \in I \mid V_i \cap A \neq \emptyset \}, \quad (1)$$

$$W := \bigcup_{j \in J} V_j. \quad (2)$$

Then $A \subset W$. Since each $V_j$ is contained in some $U_p$ we have

$$q \notin \bigcup_{j \in J} V_j = W.$$

(ii) Let $A, B \subset X$ be disjoint closed sets. Arguing as in (i) with $B$ in place of $q$ shows that there exist disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V. \quad \square$

**Definition 4** A topological space is called $\sigma$-compact if it is a countable union of compact subsets.
Theorem 1 Let $X$ be a topological space in which every point $p$ has a neighbourhood $U_p$ which is second countable and precompact (i.e. its closure is compact). Then among the following statements the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ hold:

(i) $X$ is second countable.
(ii) $X$ is $\sigma$-compact.
(iii) $X$ is paracompact.
(iv) Every component of $X$ is second countable.

Proof. $(i) \Rightarrow (ii)$: Let $X$ be second countable. Then every open cover of $X$ has a countable subcover. Applying this to the open cover $\{U_p\}_{p \in X}$ we see that there is a sequence $p_1, p_2, \ldots$ in $X$ such that

$$X = \bigcup_{k=1}^{\infty} U_{p_k}.$$  

Since $U_p$ is compact for each $p$ it follows that $X$ is $\sigma$-compact.

$(ii) \Rightarrow (iii)$: Let $X$ be $\sigma$-compact, say $X = \bigcup_{n=1}^{\infty} K_n$, where each $K_n$ is compact. We first find a sequence

$$V_0 \subset V_1 \subset V_2 \subset \ldots$$

of precompact open subsets of $X$ whose union is all of $X$, such that $V_j \subset V_{j+1}$ for all $j$. Let $V_0 := \emptyset$. After $V_0, \ldots, V_j$ have been chosen, note that $K_j \cup V_j$ is compact, so there are finitely many points $p_1, \ldots, p_m \in X$ such that

$$K_j \cup V_j \subset U_{p_1} \cup \ldots \cup U_{p_m}.$$  

Set $V_{j+1} := U_{p_1} \cup \ldots \cup U_{p_m}$.

Now let $\{W_i\}_{i \in I}$ be any open cover of $X$. Each set $\overline{V}_k \setminus V_{k-1}$ is compact and is therefore contained in $\bigcup_{i \in I_k} W_i$ for some finite subset $I_k \subset I$. Then

$$\{W_i \setminus \overline{V}_{k-2}\}_{k \in N, i \in I_k}$$

is a locally finite, open refinement of $\{W_i\}_{i \in I}$.

$(iii) \Rightarrow (iv)$: Suppose $X$ is paracompact and non-empty, and let $Y$ be a component of $X$. We will show that $Y$ is second countable. Since $Y$ is closed in $X$, Proposition 1 says that $Y$ is paracompact. Let $\{V_\alpha\}_{\alpha \in I}$ be a locally finite, open refinement of the open cover $\{U_p \cap Y\}_{p \in Y}$ of $Y$. Each $V_\alpha$
is compact and can therefore be covered by finitely many $V_{\alpha'}$, $\alpha' \in I$. We can now find a sequence of finite subsets $J_0 \subset J_1 \subset J_2 \subset \cdots \subset I$ such that $J_0 = \{\alpha_0\}$ for some $\alpha_0$ with $V_{\alpha_0}$ non-empty, and such that for each $k \geq 0$ one has

$$\bigcup_{\alpha \in J_k} V_{\alpha} \subset \bigcup_{\beta \in J_{k+1}} V_{\beta}.$$  

Set $J := \bigcup_k J_k$ and

$$Z := \bigcup_{\alpha \in J} V_{\alpha}.$$  

Then $\overline{Z} = Z$ by Lemma 1, so $Z$ is both open and closed. Because $Z$ is non-empty and $Y$ is connected we must have $Z = Y$. But then $Y$ is a countable union of subspaces $V_{\alpha}$ each of which is second countable, hence $Y$ is second countable. \qed

**Lemma 2 (Shrinking lemma)** Let $X$ be a paracompact Hausdorff space and $\{U_i\}_{i \in I}$ an open cover of $X$. Then there exists an open cover $\{V_i\}_{i \in I}$ of $X$ such that

$$V_i \subset U_i \quad \text{for all } i \in I.$$  

**Proof.** By Proposition 2 the space $X$ is normal, so each $p \in U_i$ has a neighbourhood $W_{i,p}$ such that $\overline{W}_{i,p} \subset U_i$. The collection of all such sets $W_{i,p}$ with $i \in I$ and $p \in U_i$ is an open cover of $X$ which has a locally finite, open refinement $\{Z_j\}_{j \in J}$. Choose a map $r : J \to I$ such that $Z_j \subset U_{r(j)}$ for each $j \in J$, and set

$$V_i := \bigcup_{r(j) = i} Z_j.$$  

Then by Lemma 1 we have

$$V_i = \bigcup_{r(j) = i} Z_j \subset U_i.$$

\qed