Manifolds, V2018

Problem sheet 2, to be discussed on Monday the 29th January 2018.

Problem 1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $N := (0, \ldots, 0, 1) \in S^n$ the "north pole". *Stereographic projection*

$$\tau_+ : S^n \setminus \{N\} \to \mathbb{R}^n$$

is characterized by the fact that for every point $x \in S^n \setminus \{N\}$ the line in $\mathbb{R}^{n+1}$ through N and x passes through $(\tau_+(x), 0) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. Stereographic projection from the "south pole" $-N$,

$$\tau_- : S^n \setminus \{-N\} \to \mathbb{R}^n$$

is defined similarly. (Note that $\tau_-(-x) = -\tau_+(x)$.)

(i) Find explicit formulas for τ_+ and τ_- and show that they are diffeomorphisms.

(ii) Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2 by comparing the transition map $\tau_- \circ (\tau_+)^{-1}$ for $n = 2$ with the transition map between the standard two charts on $\mathbb{CP}^1$.

Problem 2. Let M be a smooth n–manifold and $\mathrm{Diff}(M)$ the group of diffeomorphisms $M \to M$. Let G be a finite group with identity element e and

$$\phi : G \to \mathrm{Diff}(M)$$

a group homomorphism. Then G acts on M from the left by

$$g \cdot a := \phi(g)(a)$$

for $g \in G$, $a \in M$. Suppose this action is *free*, i.e. if $g \neq e$ then $g \cdot a \neq a$ for all a. Show that the quotient space M/G is a topological n–manifold and has a unique smooth structure such that the projection map $M \to M/G$ is a local diffeomorphism. (In general, a smooth map $f : M \to N$ is called a local diffeomorphism if every point in M has a neighbourhood which is mapped diffeomorphically by f onto an open set in N.)