Manifolds, V2018

Problem sheet 7, to be discussed on Monday the 12th March 2018.

Problem 1. Let X be a smooth vector field on a manifold M. For every $p \in M$ let

$$\phi_p : (a_p, b_p) \to M$$

be the maximal integral curve of X starting at p (i.e. $\phi_p(0) = p$). Here,

$$-\infty \leq a_p < 0 < b_p \leq \infty,$$

and "maximal" means that if J is any open interval containing 0 and if $\alpha : J \to M$ is an integral curve of X starting at p then $J \subset (a_p, b_p)$ and $\alpha = \phi_p|_J$. Let $\mathcal{D} \subset \mathbb{R} \times M$ be the subset given by

$$\mathcal{D} := \bigcup_{p \in M} (a_p, b_p) \times \{p\}.$$

The map

$$\phi : \mathcal{D} \to M, \quad (t, p) \mapsto \phi_p(t)$$

is called the (maximal) *flow* of X. Prove the following.

(i) $\phi(s, \phi(t, p)) = \phi(s + t, p)$ whenever both sides are defined.

(ii) $\mathcal{D}$ is open in $\mathbb{R} \times M$ and ϕ is smooth.
Hint: For each $p \in M$ let I_p be the set of all $t \in \mathbb{R}$ such that there exist an open interval J containing 0 and t and a neighbourhood U of p such that $J \times U \subset \mathcal{D}$ and $\phi|_{J \times U}$ is smooth. Use the uniqueness of integral curves and the existence of local flows to show that $\inf I_p = a_p$ and $\sup I_p = b_p$.

Problem 2. Let

$$F : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2, \quad (r, \theta) \mapsto (r \cos \theta, r \sin \theta).$$

Because F is a local diffeomorphism, every vector field Z on $\mathbb{R}^2$ is F–related to a unique vector field F^*Z on $(0, \infty) \times \mathbb{R}$. Compute F^*Z when

$$Z = (x^2 + y^2) \frac{\partial}{\partial x}.$$

Give the answer in the form

$$F^*Z = u(r, \theta) \frac{\partial}{\partial r} + v(r, \theta) \frac{\partial}{\partial \theta}$$

where u and v are real-valued functions on $(0, \infty) \times \mathbb{R}$.