Manifolds, V2018

Problem sheet 9, to be discussed on Monday the 30th April 2018.

Problem 1. Let M be a connected, oriented manifold and $\tau : M \to M$ a smooth, fixed-point free involution (i.e. $f(p) \neq p$ and $f(f(p)) = p$ for all $p \in M$). Let M/τ be the quotient manifold obtained by identifying p with $\tau(p)$ for all $p \in M$. Show that M/τ is orientable if and only if τ preserves orientation.

Problem 2. The *Möbius band* is the quotient

$$B := (S^1 \times (-1, 1))/\tau,$$

where $\tau(z, t) = (-z, -t)$. Show that B is not orientable.

Problem 3. Let M be an n–manifold (without boundary). By a *k–dimensional submanifold-with-boundary* of M we mean a subspace $S \subset M$ such that for every $p \in S$ there is a chart (U, ϕ) on M such that $p \in U$ and

$$\phi(U \cap S) = \phi(U) \cap (\mathbb{H}^k \times \{0\}) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

In other words, if $\phi = (x^1, \ldots, x^n)$ then $U \cap S$ is the locus in U where $x^k \geq 0$ and $x^i = 0$ for $i = k + 1, \ldots, n$. Show that S, with the subspace topology, then has a canonical structure of a k–dimensional manifold-with-boundary.

Problem 4. Let M be an n–manifold and $f : M \to \mathbb{R}$ a smooth function. Let $a \in \mathbb{R}$ be a regular value of f. Show that $f^{-1}(-\infty, a]$ is an n–dimensional submanifold-with-boundary of M.