MAT4520 - Manifolds, Spring 2018

Compulsory assignment, to be returned via Devilry by
Tuesday the 24th April at 12:30.

Problem 1. Let a_1, a_2, a_3 be integers, each greater than one, and let V be the set of points $(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\}$ such that

$$(z_1)^{a_1} + (z_2)^{a_2} + (z_3)^{a_3} = 0.$$

(i) Show that V is a regular submanifold of $\mathbb{C}^3 \setminus \{0\}$.

(ii) For any $r > 0$ let

$$S^5_r := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \ : \ |z_1|^2 + |z_2|^2 + |z_3|^2 = r^2\}$$

be the r–sphere in $\mathbb{C}^3$. Show that V intersects S^5_r transversely and compute the dimension of $Y := V \cap S^5_r$.

Hint: Observe that for any $z = (z_1, z_2, z_3) \in V$ the derivative of the function

$$f(z) := (z_1)^{a_1} + (z_2)^{a_2} + (z_3)^{a_3}$$

vanishes on the vector $(z_1/a_1, z_2/a_2, z_3/a_3)$. If $z \in Y$, check whether that vector is tangent to S^5_r.

Problem 2. The aim of this problem is to construct an immersion of the Klein bottle into S^3 and an embedding into $S^3 \times \mathbb{R}$. We will regard S^1 as the unit circle in the complex plane and S^3 as the unit sphere in $\mathbb{C}^2$.

(i) Show that the map

$$F : S^1 \times S^1 \to \mathbb{C}^2, \quad (w, z) \mapsto (\mathrm{Re}(z)w^2, \mathrm{Im}(z)w)$$

is an immersion which in fact takes values in S^3.

(ii) The *Klein bottle* Σ is the quotient of $S^1 \times S^1$ obtained by identifying (w, z) with $(-w, \bar{z})$ for every $(w, z) \in S^1 \times S^1$. Because the map $(w, z) \mapsto (-w, \bar{z})$ is a fix-point free involution of $S^1 \times S^1$, the space Σ is a topological 2–manifold with a unique smooth structure such that the projection $\pi : S^1 \times S^1 \to \Sigma$ is a local diffeomorphism. Show that there exists an immersion

$$G : \Sigma \to S^3$$

such that $G \circ \pi = F$.

(iii) Show that there are disjoint embedded circles C_1 and C_2 in Σ, each of which is mapped diffeomorphically by G onto the same circle in S^3, such that G is injective on $\Sigma \setminus (C_1 \cup C_2)$.

(iv) Prove that there is an embedding $H : \Sigma \to S^3 \times \mathbb{R}$ such that

$$H \circ \pi(w, z) = (\mathrm{Re}(z)w^2, \mathrm{Im}(z)w, \mathrm{Re}(z)).$$

Problem 3. Fix a natural number n and set

$$G := \mathrm{GL}(n+1, \mathbb{C}), \quad \mathfrak{g} := \mathrm{gl}(n+1, \mathbb{C}),$$

so that $\mathfrak{g}$ is the Lie algebra of the Lie group G. Recall that the projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a submersion.

(i) Show that the map

$$G \times \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}, \quad (A, v) \mapsto Av$$

descends to a smooth map $\psi : G \times \mathbb{CP}^n \to \mathbb{CP}^n$. To simplify notation we write $A \cdot p := \psi(A, p)$. In this notation, the requirement on ψ is that

$$A \cdot \pi(v) = \pi(Av).$$

(ii) For any $C \in \mathfrak{g}$ let X_C be the vector field on $\mathbb{CP}^n$ given by

$$(X_C)(p) := \left.\frac{d}{dt}\right|_0 (I + tC) \cdot p,$$

where $I \in G$ denotes the identity matrix. Show that X_C is smooth.

(iii) Show that the zeros of X_C are precisely the points of the form $\pi(v)$ where $v \in \mathbb{C}^{n+1}$ is an eigenvector of C.

(iv) For $A \in G$, $p \in \mathbb{CP}^n$ set

$$p \cdot A := \psi(A^{-1}, p),$$

so that $(p \cdot A) \cdot B = p \cdot (AB)$. For any $C \in \mathfrak{g}$ let Y_C be the vector field on $\mathbb{CP}^n$ given by

$$(Y_C)(p) := \left.\frac{d}{dt}\right|_0 p \cdot (I + tC).$$

Show that $Y_p = -X_p$.

(v) Show that for $C, D \in \mathfrak{g}$ the Lie bracket of the vector fields Y_C, Y_D is given by

$$[Y_C, Y_D] = Y_{[C,D]}.$$