

INTRODUCTION TO MAT4520/MAT9520 - MANIFOLDS

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A topological n -manifold is a (second-countable, Hausdorff) space M that is locally homeomorphic to $\mathbb{R}^n$. This means that each point $p \in M$ has a neighborhood U that is homeomorphic to an open subset of $\mathbb{R}^n$.

$$\begin{array}{c} p \in U \subset M \\ \downarrow x \cong \\ x(p) \in x(U) \subset \mathbb{R}^n \end{array}$$

We call (U, x) a coordinate chart around p .

Manifolds are the well-behaved shapes of geometric topology. They can arise as the space of configurations of a mechanical system, as the spacetime of the universe, or as the symmetry groups underlying the standard model of particle physics.

Less well-behaved spaces include manifolds with boundary, which are locally homeomorphic to $(-\infty, 0] \times \mathbb{R}^{n-1}$, manifolds with corners, or manifolds with singularities.

A key mathematical challenge is to classify all manifolds of a particular type, e.g. all compact, connected n -dimensional manifolds. It is typically difficult to do this without imposing some additional structure on the naked manifold.

The continuous functions $f: U \rightarrow \mathbb{R}$ form a commutative ring, denoted $C(U)$, which is isomorphic to the ring $C(x(U))$ of continuous functions $g: x(U) \rightarrow \mathbb{R}$ on the open subset $x(U) \subset \mathbb{R}^n$. (Let $g = fx^{-1}$.) If $x(U) = \mathbb{R}^n$, this is $C(\mathbb{R}^n)$.

For some functions $f: M \rightarrow \mathbb{R}$ there are many values $a \in \mathbb{R}$ such that the level set

$$N = f^{-1}(a) = \{p \in M \mid f(p) = a\}$$

is an $(n - 1)$ -dimensional submanifold of M . We may call these a regular values of f , or say that f is transversal to a , and call N a regular level set. However, there are continuous functions f with no such regular values. In order to use level sets for manifold topology we therefore need to know how to avoid such bad functions.

One approach to this is to narrow down the allowable coordinate charts (U, x) of M so as to let us make sense of when a function $f: U \rightarrow \mathbb{R}$ is piecewise linear, namely when $g = fx^{-1}: x(U) \rightarrow \mathbb{R}$ is piecewise linear on $x(U) \subset \mathbb{R}^n$. We then call M a piecewise linear manifold. These admit models as simplicial complexes, known as combinatorial manifolds. This was the approach taken by Henri Poincaré in the 1890s.

Another approach is to instead make sense of when a function $f: U \rightarrow \mathbb{R}$ is r times continuously differentiable, for some degree of regularity $1 \leq r \leq \infty$. We then call M a C^r differentiable manifold. Then a is a regular value if for each $p \in N$ the derivative $df_p: T_p M \rightarrow T_a \mathbb{R}$ is nonzero. In this course we will mainly

consider smooth, i.e., C^∞ differentiable, manifolds. This approach was taken by Marston Morse in the 1920s. See John Milnor's books on "Morse Theory" and "The h -Cobordism Theorem".

Moreover, if all values in an interval $[a, b] \subset \mathbb{R}$ are regular, it is often the case that

$$W = f^{-1}([a, b]) = \{p \in M \mid a \leq f(p) \leq b\} \cong N \times [a, b]$$

is an n -dimensional manifold with boundary. Then $f^{-1}(a) = N \cong f^{-1}(b)$, so the isomorphism class of the regular level set does not change from a to b .

The generalized Poincaré conjecture in dimensions $n \geq 5$ was proved in the 1960s by Stephen Smale using this idea. An n -manifold M^n that is homotopy equivalent to the unit sphere S^n in $\mathbb{R}^{n+1}$ is shown to admit a (piecewise linear or smooth) function $f: M^n \rightarrow \mathbb{R}$ with only two critical points: a minimum p and a maximum q . For a a little larger than $f(p)$ and b a little smaller than $f(q)$ the division of M into the submanifolds $f^{-1}((-\infty, a])$, $f^{-1}([a, b])$ and $f^{-1}([b, \infty))$ is then homeomorphic to the division of S^n into $D^n \times \{a\}$, $\partial D^n \times [a, b]$ and $D^n \times \{b\}$.

In the 1930s and 1940s, Lev Pontryagin instead considered realizing suitable n -dimensional manifolds as the level sets $M^n = f^{-1}(a)$ for regular values $a \in S^k$ of maps $f: S^{k+n} \rightarrow S^k$. If we make a different choice of regular value a , or deform f in a parametrized family called a homotopy, we get a different level set manifold M' . In general, two such choices M and M' will not be isomorphic (e.g. as smooth manifolds), but they will satisfy a weaker equivalence relation called cobordism. This leads to a classification of cobordism classes of suitable n -dimensional manifolds in terms of homotopy classes of maps $S^{k+n} \rightarrow S^k$ for large k .

Pontryagin first used his knowledge of 1- and 2-dimensional manifolds to determine the homotopy classes of maps $S^{k+1} \rightarrow S^k$ and $S^{k+2} \rightarrow S^k$, but in later years this classification scheme has instead been turned around, so that algebraic topological calculations of homotopy classes of maps $S^{k+n} \rightarrow S^k$, for some n , determine the cobordism classification of suitable n -manifolds, namely those that are stably framed. In the 1950s, René Thom modified Pontryagin's approach, obtaining a bijective correspondence between homotopy classes of maps $S^{k+n} \rightarrow MO_k$ for suitable spaces MO_k (called Thom complexes) and cobordism classes of all closed n -manifolds (compact, without boundary). This led to a complete calculation of these cobordism classes. For instance, each closed 3-manifold $M = \partial W$ is the boundary of a compact 4-manifold (with boundary, of course), and this makes it possible to use a choice of W to study M .

A key task at the beginning of this course is to make sense of what we mean by a tangent vector v to a (C^r for $r \geq 1$ or) smooth n -manifold M at a point $p \in M$. The collection of all of these forms an n -dimensional vector space $T_p M$, called the tangent space of M at p . Several approaches are possible:

- If $i: M \subset \mathbb{R}^N$ is a submanifold of Euclidean N -space, then the tangent vectors at $p \in M$ are the derivatives $(ic)'(0)$ of smooth curves $c: \mathbb{R} \rightarrow M$ with $c(0) = p$. Thus $T_p M \subset \mathbb{R}^N$. This definition is not intrinsic.
- In general, tangent vectors at $p \in M$ can be defined as equivalence classes of smooth curves $c: \mathbb{R} \rightarrow M$ with $c(0) = p$, thinking of the class $[c]$ as specifying the velocity vector $c'(0) \in T_p M$. This requires knowing when two curves c_1 and c_2 give equivalent tangent vectors.

- One can also declare that a tangent vector $v \in T_p M$ is defined by its image $dx_p(v) \in T_{x(p)} \mathbb{R}^n \cong \mathbb{R}^n$, for some coordinate chart (U, x) with $p \in U$. This requires comparing the images for different coordinate charts (U, x) and (V, y) with $p \in U \cap V$.
- Covariant derivation along v is a linear operator $D_v: C^\infty(M) \rightarrow \mathbb{R}$ taking a smooth function $f: M \rightarrow \mathbb{R}$ to its directional derivative $D_v f \in \mathbb{R}$. These operators all satisfy a Leibniz rule at p , in the sense that if $g: M \rightarrow \mathbb{R}$ is another smooth function, and $(f \cdot g)(q) = f(q) \cdot g(q)$ for $q \in M$ defines their pointwise product, then

$$D_v(f \cdot g) = D_v f \cdot g(p) + f(p) \cdot D_v g.$$

The rule $v \mapsto D_v$ mapping $T_p M$ to the vector space of derivations at p

$$D: C^\infty(M) \rightarrow \mathbb{R}$$

is an isomorphism. This can be taken as a definition of v and $T_p M$, making each tangent vector a particular linear operator on $C^\infty(M)$. For each neighborhood U of p , the derivations $C^\infty(U) \rightarrow \mathbb{R}$ at p give yet another equivalent model for $T_p M$.

Each differentiable map $f: M \rightarrow N$ will then induce a derivative, or best linear approximation, $df_p: T_p M \rightarrow T_{f(p)} N$, which is a linear homomorphism of real vector spaces. We say that p is a regular point for f if df_p is surjective, and that $a \in N$ is a regular value if each $p \in f^{-1}(a)$ is a regular point.

The collection of all tangent spaces $T_p M$ to a C^r manifold M can be combined to a C^{r-1} manifold

$$TM = \bigcup_{p \in M} T_p M$$

with a canonical projection $\pi: TM \rightarrow M$, such that $\pi^{-1}(p) = T_p M$. We call TM the tangent bundle of M . A vector field on M is then a map $X: M \rightarrow TM$ with $\pi X = id_M$, associating to each $p \in M$ a tangent vector $X_p \in T_p M$. Vector fields encode first order ordinary differential equations on M . A curve $c: \mathbb{R} \rightarrow M$ satisfying

$$c'(t) = X_{c(t)} \in T_{c(t)} M$$

for each t is called an integral curve for X . Higher order differentials can be expressed in terms of $T(TM)$, etc., leading to jet bundles. Working with C^∞ manifolds is convenient, since one does not need to keep track of how much regularity is retained at each stage.

A Riemannian metric on a differentiable manifold M is a family of inner products (i.e., symmetric, bilinear, positive definite pairings)

$$\langle -, - \rangle_p: T_p M \times T_p M \rightarrow \mathbb{R}$$

for $p \in M$. Let $\|v\|_p = \sqrt{\langle v, v \rangle_p}$. The length of a curve $c: [a, b] \rightarrow M$ is the integral

$$\int_a^b \|c'(t)\|_{c(t)} dt$$

and the Riemannian distance $d(p, q)$ is the length of the shortest curve from p to q . This is the framework for differential geometry.

The formal symbol dt can be meaningfully defined as a differential 1-form. More generally, a differential k -form ω on M is a family of alternating, k -multilinear homomorphisms

$$\omega_p: T_p M \times \cdots \times T_p M \rightarrow \mathbb{R},$$

and we can integrate n -forms over n -manifolds. An operation $\omega \mapsto d\omega$ called the exterior derivative takes k -forms to $(k+1)$ -forms, and satisfies $dd = 0$. In the special case when M is an open subset of $\mathbb{R}^3$, this unifies the gradient, curl and divergence from multivariable calculus. The fundamental theorem of calculus, Green's theorem, Stokes' theorem and the divergence theorem are all unified in the single result

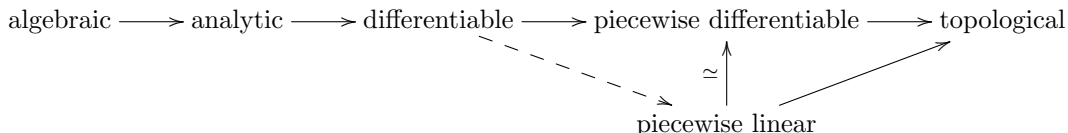
$$\int_M d\omega = \int_{\partial M} \omega.$$

Toward the end of the course we will study de Rham cohomology, which provides a measure of to what extent the converse of the curl grad = 0 and div curl = 0 formulas hold, over general smooth manifolds.

In addition to topological, piecewise linear and differentiable manifolds, there are other more strongly structured classes of manifolds. For example, if M is a (real) analytic manifold the allowable coordinate charts (U, x) are restricted so that it makes sense to say that a function $f: U \rightarrow \mathbb{R}$ is analytic. The ring $C^\omega(U)$ of such functions is then isomorphic to the ring $C^\omega(x(U))$ of functions $g: x(U) \rightarrow \mathbb{R}$ with convergent Taylor series near each point in $x(U) \subset \mathbb{R}^n$.

As another example, an algebraic manifold, or the real points of a non-singular variety, has local rings of functions $C^{alg}(U)$ isomorphic to the ring of rational functions $g: x(U) \rightarrow \mathbb{R}$, i.e., fractions $P(z_1, \dots, z_n)/Q(z_1, \dots, z_n)$ where $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ are polynomials, and $Q(z_1, \dots, z_n) \neq 0$ for $(z_1, \dots, z_n) \in x(U) \subset \mathbb{R}^n$.

Each algebraic manifold is analytic, each analytic manifold is C^r differentiable for each $1 \leq r \leq \infty$, each C^1 differentiable manifold is suitably equivalent to a piecewise linear manifold, and all of these have underlying topological manifolds.



A key difference is that algebraic and analytic functions are rigid, in the sense that if one is constant in a neighborhood of a point, then it is constant on the whole connected component of that point. On the other hand in the differentiable, piecewise linear and topological settings there exist bump functions $f: U \rightarrow \mathbb{R}$ that are constantly equal to 0 outside one neighborhood, and constantly equal to 1 within a smaller neighborhood. These bump functions, and their associated partitions of unity, allow us to glue together information in the differentiable, piecewise linear and topological categories, in a way that is not available for algebraic and analytic manifolds. The study of which manifolds of one kind can be given refined structures, and if so, in how many inequivalent ways, contains many striking results, including Milnor's discovery of smooth 7-manifolds that are homeomorphic but not diffeomorphic to S^7 .

((ETC: Relation to Morse theory and singular cohomology. Bookkeeping to pass from the known cohomology of a manifold homotopy equivalent to a sphere to the existence of a function with only two regular values.))

((ETC: Lie groups and Lie algebras.))