

SMOOTHNESS OF REAL PROJECTIVE HYPERSURFACES

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Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k , viewed as a smooth map $\mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}$. Let $Z(F) = \pi(F^{-1}(0)) \subset \mathbb{R}P^n$. Then $p \in Z(F)$ if and only if $F(a) = 0$, for any choice of a with $\pi(a) = p$.

$$\begin{array}{ccc}
 F^{-1}(0) & \longrightarrow & \mathbb{R}^{n+1} - \{0\} \xrightarrow{F} \mathbb{R} \\
 \downarrow & & \downarrow \pi \\
 Z(F) & \longrightarrow & \mathbb{R}P^n \\
 \uparrow & & \uparrow \\
 Z(F) \cap V_i & \longrightarrow & V_i \\
 \downarrow & & \downarrow y_i \\
 S_i & \longrightarrow & \mathbb{R}^n
 \end{array}$$

Lemma: The hypersurface $Z(F)$ is a codimension 1 regular (smooth) submanifold of $\mathbb{R}P^n$ if the $(n+1)$ partial derivatives

$$\frac{\partial F}{\partial x_0}(a), \dots, \frac{\partial F}{\partial x_n}(a)$$

are not all simultaneously zero for any $a \in F^{-1}(0)$.

Proof: For each $0 \leq i \leq n$ let $V_i \subset \mathbb{R}P^n$ be the open subset of points $[a_0 : \dots : a_i : \dots : a_n]$ with $a_i \neq 0$, and let $y_i: V_i \rightarrow \mathbb{R}^n$ be the smooth coordinate with inverse

$$\begin{aligned}
 y_i^{-1}(r^1, \dots, r^n) &= [r^1 : \dots : r^i : 1 : r^{i+1} : \dots : r^n] \\
 &= \pi(r^1, \dots, r^i, 1, r^{i+1}, \dots, r^n).
 \end{aligned}$$

We show that $Z(F) \cap V_i$ is a codimension 1 regular submanifold of V_i , for each i . Each $p \in Z(F)$ lies in V_i for some i , and an adapted chart at p in V_i relative to $Z(F) \cap V_i$ is then also an adapted chart at p in $\mathbb{R}P^n$ relative to $Z(F)$.

Since y_i is a diffeomorphism, it suffices to prove that $S_i = y_i(Z(F) \cap V_i)$ is a codimension 1 regular submanifold of $\mathbb{R}^n$. Here $(r^1, \dots, r^n) \in S_i$ if and only if $F(r^1, \dots, r^i, 1, r^{i+1}, \dots, r^n) = 0$. Let $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ be given by

$$f_i(r^1, \dots, r^n) = F(r^1, \dots, r^i, 1, r^{i+1}, \dots, r^n)$$

so that $S_i = f_i^{-1}(0)$ is the zero set of f_i . By the regular level set theorem, it suffices to check that 0 is a regular value of f_i . Suppose, in order to obtain a contradiction, that $s = (r^1, \dots, r^n) \in S_i$ is a critical point for f_i . Then

$$\frac{\partial f_i}{\partial r^1}(s), \dots, \frac{\partial f_i}{\partial r^n}(s)$$

are all zero. This is equivalent to the n partial derivatives

$$\frac{\partial F}{\partial x_0}(a), \dots, \frac{\partial F}{\partial x_{i-1}}(a), \frac{\partial F}{\partial x_{i+1}}(a), \dots, \frac{\partial F}{\partial x_n}(a)$$

all being zero at $a = (r^1, \dots, r^i, 1, r^{i+1}, \dots, r^n) \in F^{-1}(0)$. Euler's formula (from Problem 9.6)

$$\sum_{j=0}^n x_j \frac{\partial F}{\partial x_j}(a) = kF(a),$$

obtained by partial differentiation at $t = 1$ of the homogeneity condition

$$F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n),$$

would then imply the vanishing $\partial F / \partial x_i(a) = 0$ of the last partial derivative, and this contradicts the hypothesis that at least one of

$$\frac{\partial F}{\partial x_0}(a), \dots, \frac{\partial F}{\partial x_n}(a)$$

is nonzero at $a \in F^{-1}(0)$.

Hence the hypothesis ensures that each $s \in S_i$ is a regular point, that 0 is a regular value of f_i , that $S_i = f_i^{-1}(0) \subset \mathbb{R}^n$ is a regular submanifold, and that $y_i^{-1}(S_i) = Z(F) \cap V_i \subset V_i$ is a regular submanifold. Since this holds for each i , it follows that $Z(F) \subset \mathbb{R}P^n$ is a regular submanifold and, in particular, is smooth. Q.E.D.

Example: Consider parameters a, b . The affine cubic curve $y^2 = x^3 + ax + b$ in the xy -plane extends to the projective curve in the projective plane consisting of points $[x : y : z]$ with (x, y, z) satisfying the equation

$$y^2 z = x^3 + axz^2 + bz^3.$$

This is $Z(F)$ for the homogeneous polynomial $F(x, y, z) = y^2 z - x^3 - axz^2 - bz^3$ of degree $k = 3$. If $x^3 + ax + b$ has three distinct (real) roots, this is a smooth curve in $\mathbb{R}P^2$, homeomorphic to two copies of S^1 . If $x^3 + ax + b$ has one real and two complex roots, then this is a smooth curve homeomorphic to S^1 . If $x^3 + ax + b$ has one simple and one double real root, or a triple real root, then $Z(F)$ is not a regular (smooth) curve.

For each 2-dimensional linear subspace $P \subset \mathbb{R}^{n+1}$, the image $L = \pi(P - \{0\}) \subset \mathbb{R}P^n$ is called a projective line.

Lemma: Let $F \neq 0$. The intersection $Z(F) \cap L$ consists of at most k points.

Lemma: Let $Z(F)$ be smooth. If $Z(F)$ and L meet transversely, then the number of intersection points is congruent to $k \pmod{2}$.

The transversality condition asks that for each intersection point $p \in Z(F) \cap L$ the tangent spaces $T_p Z(F)$ and $T_p L$ together span $T_p \mathbb{R}P^n$, i.e., that $T_p L \not\subset T_p Z(F)$.

Allowing complex, in place of real, coordinates, we can view the polynomial F as a smooth map $F_{\mathbb{C}}: \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}$, and let $Z_{\mathbb{C}}(F) = \pi(F_{\mathbb{C}}^{-1}(0)) \subset \mathbb{C}P^n$. Then $Z_{\mathbb{C}}(F)$ is a complex codimension 1 (= real codimension 2) regular (smooth) submanifold of $\mathbb{C}P^n$. For each complex projective line $L_{\mathbb{C}} \subset \mathbb{C}P^n$ the intersection $Z_{\mathbb{C}}(F) \cap L_{\mathbb{C}}$ consists of k points, counted with multiplicities. If $Z_{\mathbb{C}}(F)$ and $L_{\mathbb{C}}$ meet transversely, then the number of intersection points is exactly k , i.e., each point has multiplicity one.

Theorem (Nash (1952), Tognoli (1973)): Every smooth, compact manifold M is diffeomorphic to [a connected component of] a smooth real algebraic set $X \subset \mathbb{R}P^N$.