

**SAMPLE SOLUTIONS TO MANDATORY ASSIGNMENT  
FOR MAT4520 MANIFOLDS SPRING 2022**

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**Problem 1:** Each  $U_{i,\epsilon}$  is open in  $S^n$ , since  $\epsilon a_i > 0$  is an open condition. The sum  $a_0^2 + \cdots + a_{i-1}^2 + a_{i+1}^2 + \cdots + a_n^2$  can be written as  $1 - a_i^2$  with  $\epsilon a_i > 0$  if and only if it lies in  $[0, 1)$ , so the image

$$\begin{aligned} x_{i,\epsilon}(U_{i,\epsilon}) &= \{(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in \mathbb{R}^n \mid a_0^2 + \cdots + a_{i-1}^2 + a_{i+1}^2 + \cdots + a_n^2 \in [0, 1)\} \\ &= \{(b_1, \dots, b_n) \mid b_1^2 + \cdots + b_n^2 < 1\} = B^n \end{aligned}$$

is the open unit ball in  $\mathbb{R}^n$ . For brevity, we write  $|b|^2 = b_1^2 + \cdots + b_n^2$  for the square norm of  $b = (b_1, \dots, b_n)$ . The functions  $x_{i,\epsilon}: U_{i,\epsilon} \rightarrow B^n$  and  $x_{i,\epsilon}^{-1}: B^n \rightarrow U_{i,\epsilon}$ , with targets the subspaces  $B^n \subset \mathbb{R}^n$  and  $U_{i,\epsilon} \subset S^n \subset \mathbb{R}^{n+1}$ , are given by

$$\begin{aligned} x_{i,\epsilon}(a_0, \dots, a_n) &= (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \\ x_{i,\epsilon}^{-1}(b_1, \dots, b_n) &= (b_1, \dots, b_i, \epsilon\sqrt{1 - |b|^2}, b_{i+1}, \dots, b_n). \end{aligned}$$

These are mutually inverse, and both are continuous since the argument  $1 - |b|^2$  to the square root is positive for  $(b_1, \dots, b_n) \in B^n$ . Hence  $x_{i,\epsilon}$  is a homeomorphism from the open subset  $U_{i,\epsilon}$  of  $S^n$  to the open subset  $x_{i,\epsilon}(U_{i,\epsilon}) = B^n$  of  $\mathbb{R}^n$ . The collection  $\mathcal{U}$  covers  $S^n$ , since for each  $a \in S^n$  some  $a_i$  is nonzero, and if  $a_i > 0$  then  $a \in U_{i,+}$ , while if  $a_i < 0$  then  $a \in U_{i,-}$ . Hence  $\mathcal{U}$  forms a (topological) atlas exhibiting  $S^n$  as a (topological)  $n$ -manifold.

To check that this is a  $C^\infty$  atlas, exhibiting  $S^n$  as a  $n$ -dimensional  $C^\infty$  manifold, we show that each transition function

$$x_{j,\eta} x_{i,\epsilon}^{-1}: x_{i,\epsilon}(U_{i,\epsilon} \cap U_{j,\eta}) \longrightarrow x_{j,\eta}(U_{i,\epsilon} \cap U_{j,\eta})$$

with  $i, j \in \{0, \dots, n\}$ ,  $\epsilon, \eta \in \{+, -\}$  is a  $C^\infty$  map between open subsets of  $\mathbb{R}^n$ . (As usual,  $U_{i,\epsilon} \cap U_{j,\eta}$  is open in  $U_{i,\epsilon}$ , so  $x_{i,\epsilon}(U_{i,\epsilon} \cap U_{j,\eta})$  is open in  $x_{i,\epsilon}(U_{i,\epsilon}) = B^n$ , hence also in  $\mathbb{R}^n$ .) Here

$$x_{j,\eta} x_{i,\epsilon}^{-1}(b_1, \dots, b_n) = (b_1, \dots, b_i, \epsilon\sqrt{1 - |b|^2}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_n)$$

for  $i < j$ ,

$$x_{j,\eta} x_{i,\epsilon}^{-1}(b_1, \dots, b_n) = (b_1, \dots, b_n)$$

for  $i = j$ , and

$$x_{j,\eta} x_{i,\epsilon}^{-1}(b_1, \dots, b_n) = (b_1, \dots, b_j, b_{j+2}, \dots, b_i, \epsilon\sqrt{1 - |b|^2}, b_{i+1}, \dots, b_n)$$

for  $i > j$ . These are  $C^\infty$  functions from  $x_{i,\epsilon}(U_{i,\epsilon} \cap U_{j,\eta})$  to  $x_{j,\eta}(U_{i,\epsilon} \cap U_{j,\eta}) \subset x_{j,\eta}(U_{j,\eta}) = B^n \subset \mathbb{R}^n$ , because the argument  $1 - |b|^2$  to the square root function is positive for  $(b_1, \dots, b_n) \in x_{i,\epsilon}(U_{i,\epsilon} \cap U_{j,\eta}) \subset B^n$ , and  $\sqrt{1 - t}$  is a smooth function of  $t$  for  $t < 1$ .

**Problem 2:** Each  $V_i$  is open in  $\mathbb{R}P^n$ , since  $\pi^{-1}(V_i) = U_{i,+} \cup U_{i,-}$  is open in  $S^n$ . A similar argument shows that  $\pi|_{U_{i,+}}: U_{i,+} \rightarrow V_i$  is a homeomorphism. Hence

$$y_i = x_{i,+} \circ (\pi|_{U_{i,+}})^{-1}$$

is a homeomorphism from  $V_i$  to  $y_i(V_i) = x_{i,+}(U_{i,+}) = B^n$ , which is open in  $\mathbb{R}^n$ . The collection  $\mathcal{V}$  covers  $\mathbb{R}P^n$  since each  $[a] \in \mathbb{R}P^n$  has the form  $\pi(a)$  for some  $a \in S^n$ , with  $a_i \neq 0$  for some  $i \in \{0, \dots, n\}$ . If  $a_i < 0$  we replace  $a$  by  $-a$  without changing  $\pi(a)$ . Then  $a_i \in U_{i,+}$  and  $[a] \in \pi(U_{i,+}) = V_i$ . Hence  $\mathcal{V}$  is a (topological) atlas presenting  $\mathbb{R}P^n$  as a (topological)  $n$ -manifold.

To check that  $\mathcal{V}$  is a  $C^\infty$  atlas, we show that each transition function

$$y_j y_i^{-1}: y_i(V_i \cap V_j) \longrightarrow y_j(V_i \cap V_j)$$

with  $i, j \in \{0, \dots, n\}$  is a  $C^\infty$  map. Note that

$$y_i^{-1}(b_1, \dots, b_n) = [b_1 : \cdots : b_i : \sqrt{1 - |b|^2} : b_{i+1} : \cdots : b_n]$$

for  $(b_1, \dots, b_n) \in B^n$  and

$$y_j([a_0 : \dots : a_n]) = \text{sgn}(a_j)(a_0, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$$

for  $[a_0 : \dots : a_n] \in V_j$ , where  $\text{sgn}(a_j) = +1$  if  $a_j > 0$  and  $\text{sgn}(a_j) = -1$  if  $a_j < 0$ . We have  $[a] \in V_i \cap V_j$  if and only if  $a_i \neq 0$  and  $a_j \neq 0$ , so

$$y_i(V_i \cap V_j) = \begin{cases} \{(b_1, \dots, b_n) \in B^n \mid b_j \neq 0\} & \text{for } i < j, \\ B^n & \text{for } i = j, \\ \{(b_1, \dots, b_n) \in B^n \mid b_{j+1} \neq 0\} & \text{for } i > j. \end{cases}$$

Hence

$$y_j y_i^{-1}(b_1, \dots, b_n) = \text{sgn}(b_j)(b_1, \dots, b_i, \sqrt{1 - |b|^2}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_n)$$

for  $i < j$ ,

$$y_j y_i^{-1}(b_1, \dots, b_n) = (b_1, \dots, b_n)$$

for  $i = j$ , and

$$y_j y_i^{-1}(b_1, \dots, b_n) = \text{sgn}(b_{j+1})(b_1, \dots, b_j, b_{j+2}, \dots, b_i, \sqrt{1 - |b|^2}, b_{i+1}, \dots, b_n)$$

for  $i > j$ . These are all  $C^\infty$  on  $y_i(V_i \cap V_j)$ , since  $\text{sgn}(t)$  is a smooth function of  $t$  for  $t \neq 0$  and  $\sqrt{1 - t}$  is a smooth function of  $t$  for  $t < 1$ .

**Problem 3:** We first show that  $\pi$  restricts to a diffeomorphism  $\pi|_{U_{i,\epsilon}}: U_{i,\epsilon} \rightarrow V_i$  for each  $i \in \{0, \dots, n\}$  and  $\epsilon \in \{+, -\}$ . In coordinates,  $y_i \circ \pi|_{U_{i,\epsilon} \circ x_{i,\epsilon}^{-1}}: B^n \rightarrow B^n$  is given by the composite

$$\begin{aligned} b = (b_1, \dots, b_n) &\xrightarrow{x_{i,\epsilon}^{-1}} (b_1, \dots, b_i, \epsilon\sqrt{1 - |b|^2}, b_{i+1}, \dots, b_n) \\ &\xrightarrow{\pi|_{U_{i,\epsilon}}} [b_1 : \dots : b_i : \epsilon\sqrt{1 - |b|^2} : b_{i+1} : \dots : b_n] \\ &= [\epsilon b_1 : \dots : \epsilon b_i : \sqrt{1 - |b|^2} : \epsilon b_{i+1} : \dots : \epsilon b_n] \\ &\xrightarrow{y_i} (\epsilon b_1, \dots, \epsilon b_i, \epsilon b_{i+1} : \dots : \epsilon b_n) = \epsilon b. \end{aligned}$$

This composite map  $B^n \rightarrow B^n$  is the restriction of a linear map (plus or minus the identity of  $\mathbb{R}^n$ ), hence is  $C^\infty$ . Moreover, it is its own inverse, hence is a diffeomorphism. The coordinate maps  $x_{i,\epsilon}: U_{i,\epsilon} \rightarrow B^n$  and  $y_i: V_i \rightarrow B^n$  are diffeomorphisms, which implies that the composite

$$\pi|_{U_{i,\epsilon}} = y_i^{-1} \circ (y_i \circ \pi|_{U_{i,\epsilon}} \circ x_{i,\epsilon}^{-1}) \circ x_{i,\epsilon}$$

is also a diffeomorphism, as claimed. Since the  $U_{i,\epsilon}$  form an open cover of  $S^n$ , this proves that  $\pi$  is a local diffeomorphism.

**Problem 4:** [Correction: The left hand side in the display should be  $gx_{i,\epsilon}^{-1}(b_1, \dots, b_n)$ , not  $g(b_1, \dots, b_n)$ .]

We apply  $g(a_0, \dots, a_n) = \sum_{j=1}^n j a_j^2 = \sum_{j=0}^n j a_j^2$  to  $x_{i,\epsilon}^{-1}(b_1, \dots, b_n) = (b_1, \dots, b_i, \epsilon\sqrt{1 - |b|^2}, b_{i+1}, \dots, b_n)$  to get

$$\begin{aligned} gx_{i,\epsilon}^{-1}(b_1, \dots, b_n) &= \sum_{j=0}^{i-1} j b_{j+1}^2 + i \left( \epsilon\sqrt{1 - |b|^2} \right)^2 + \sum_{j=i+1}^n j b_j^2 \\ &= \sum_{j=1}^i (j-1) b_j^2 + i - \sum_{j=1}^i i b_j^2 - \sum_{j=i+1}^n i b_j^2 + \sum_{j=i+1}^n j b_j^2 \\ &= i - \sum_{j=1}^i (i+1-j) b_j^2 + \sum_{j=i+1}^n (j-i) b_j^2. \end{aligned}$$

**Problem 5:** The map  $g: S^n \rightarrow \mathbb{R}$  is  $C^\infty$  if (and only if) the composite  $g|_{U_{i,\epsilon}} \circ x_{i,\epsilon}^{-1}: B^n \rightarrow \mathbb{R}$  is  $C^\infty$  for each coordinate chart  $(U_{i,\epsilon}, x_{i,\epsilon})$  in the  $C^\infty$  atlas defining the  $C^\infty$  structure on  $S^n$ . For each  $i \in \{0, \dots, n\}$  and  $\epsilon \in \{+, -\}$  we showed in Problem 4 that  $g \circ x_{i,\epsilon}^{-1}$  is the restriction to the open subset  $B^n$  of a polynomial function  $\mathbb{R}^n \rightarrow \mathbb{R}$ , which is certainly  $C^\infty$ .

**Problem 6:** A point  $s \in S^n$  is a critical point of  $g$  if  $g_{*,s}: T_s S^n \rightarrow T_{g(s)} \mathbb{R}$  is not surjective, or equivalently if  $dg_s: T_s S^n \rightarrow \mathbb{R}$  is zero. If  $s \in U_{i,\epsilon}$  this is equivalent to the vanishing of

$$\frac{\partial g}{\partial x_{i,\epsilon}^j}(s) = \frac{\partial(gx_{i,\epsilon}^{-1})}{\partial b_j}(b)$$

for all  $j \in \{1, \dots, n\}$ , where  $x_{i,\epsilon}(s) = b = (b_1, \dots, b_n) \in B^n \subset \mathbb{R}^n$ . Taking partial derivatives in the formula from Problem 4, we have

$$\frac{\partial(gx_{i,\epsilon}^{-1})}{\partial b_j}(b) = \begin{cases} -2(i+1-j)b_j & \text{for } j \in \{1, \dots, i\}, \\ 2(j-i)b_j & \text{for } j \in \{i+1, \dots, n\}, \end{cases}$$

which is zero for each  $j$  if and only if  $b = (0, \dots, 0)$ . Hence  $s \in U_{i,\epsilon}$  is a critical point for  $g$  if and only if  $x_{i,\epsilon}(s) = (0, \dots, 0)$ , so that

$$s = s_{i,\epsilon} = x_{i,\epsilon}^{-1}(0, \dots, 0) = \epsilon(0, \dots, 0, 1, 0, \dots, 0).$$

The function  $g$  therefore has  $2(n+1)$  critical points  $s_{i,\epsilon}$  for  $i \in \{0, \dots, n\}$ ,  $\epsilon \in \{+, -\}$ . The critical value at  $s_{i,\epsilon}$  is  $g(s_{i,\epsilon}) = 0 + \dots + 0 + i \cdot \epsilon^2 + 0 + \dots + 0 = i$ .

**Problem 7:** To prove that  $f$  is  $C^\infty$  at a point  $p \in \mathbb{R}P^n$ , choose a point  $a \in S^n$  with  $\pi(a) = p$ . (This uses that  $\pi$  is surjective.) By Problem 3 there is a neighborhood  $U$  of  $a$  such that  $\pi(U)$  is a neighborhood of  $p$  and  $\pi|_U: U \rightarrow \pi(U)$  is a diffeomorphism. Then  $f|_{\pi(U)} = g|_U \circ (\pi|_U)^{-1}$  is a composite of  $C^\infty$  functions, by Problem 5, hence  $f$  is  $C^\infty$  on  $\pi(U)$ . Since  $p$  was arbitrary,  $f$  is  $C^\infty$ .

**Problem 8:** From  $f \circ \pi = g$  and the chain rule we have  $f_{*,\pi(a)} \circ \pi_{*,a} = g_{*,a}$  for each  $a \in S^n$ . Since  $\pi$  is a local diffeomorphism by Problem 3,  $\pi_{*,a}$  is an isomorphism, so  $\pi(a)$  is a critical point of  $f$  if and only if  $f_{*,\pi(a)} = 0$  if and only if  $g_{*,a} = 0$  if and only if  $a$  is a critical point of  $g$ . (All critical points of  $f$  arise this way, since  $\pi$  is surjective.) From Problem 6, it follows that the critical points of  $f$  are  $p_i = \pi(s_{i,\epsilon}) = [0 : \dots : 0 : 1 : 0 : \dots : 0]$  for  $i \in \{0, \dots, n\}$ . The critical values are  $f(p_i) = g(s_{i,\epsilon}) = i$ , again for  $i \in \{0, \dots, n\}$ .

**Problem 9:** [Correction: It would be more precise to insert ‘non-degenerate’ before ‘critical point’ in the problem statement.]

For each  $0 \leq i \leq n$  we prove that the critical point  $p_i = [0 : \dots : 0 : 1 : 0 : \dots : 0]$  of  $f: \mathbb{R}P^n \rightarrow \mathbb{R}$  is non-degenerate of index  $i$ . Consider the coordinate chart  $(V_i, y_i: V_i \rightarrow \mathbb{R}^n)$ , for which  $p_i \in V_i$  and  $y_i(p_i) = 0 \in \mathbb{R}^n$ . For  $a \in U_{i,+}$  we have  $[a] \in V_i$  and  $y_i([a]) = x_{i,+}(a) = b \in B^n$ . Hence  $y_i^j([a]) = b_j$  and by Problem 4 we have

$$\begin{aligned} f([a]) &= g(a) = gx_{i,+}^{-1}(b) = i - \sum_{j=1}^i (i+1-j)b_j^2 + \sum_{j=i+1}^n (j-i)b_j^2 \\ &= i + \sum_{j=1}^n c_j b_j^2 = f(p_i) + \sum_{j=1}^n c_j y_i^j([a])^2, \end{aligned}$$

where

$$c_j = \begin{cases} -(i+1-j) < 0 & \text{for } j \in \{1, \dots, i\}, \\ (j-i) > 0 & \text{for } j \in \{i+1, \dots, n\} \end{cases}$$

are all nonzero. Since each  $r \in V_i$  has the form  $[a]$  with  $a \in U_{i,+}$ , this verifies the condition for  $p_i$  to be non-degenerate. Its index is the number of negative coefficients  $c_j$ , which equals  $i$ .

**Problem 10:** The numbers  $1/2$ ,  $3/2$  and  $5/2$  are not critical values of  $f: \mathbb{R}P^3 \rightarrow \mathbb{R}$  by Problem 8. Hence they are regular values, so the preimages  $f^{-1}(1/2)$ ,  $f^{-1}(3/2)$  and  $f^{-1}(5/2)$  are all regular submanifolds of  $\mathbb{R}P^3$ , of dimension  $\dim(\mathbb{R}P^3) - \dim(\mathbb{R}) = 3 - 1 = 2$ , by the regular level set (alias regular value) theorem.

**Problem 11:** For  $a = (a_0, a_1, a_2, a_3) \in S^3$  we have  $[a] \in L_{1/2}$  if and only if  $g(a) = f([a]) = 1/2$ , so that

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1 \\ a_1^2 + 2a_2^2 + 3a_3^2 &= 1/2 \end{aligned}$$

are both satisfied. From  $1 - a_0^2 = a_1^2 + a_2^2 + a_3^2 \leq a_1^2 + 2a_2^2 + 3a_3^2 = 1/2$  we deduce  $a_0^2 \geq 1/2$ , so  $L_{1/2} \subset V_0$ . Hence  $y_0: V_0 \rightarrow B^3$  maps  $L_{1/2}$  diffeomorphically (hence homeomorphically) to

$$y_0(L_{1/2}) = \{(b_1, b_2, b_3) \in B^3 \mid b_1^2 + 2b_2^2 + 3b_3^2 = 1/2\}$$

which is an ellipsoid in  $\mathbb{R}^3$ , hence is diffeomorphic (and homeomorphic) to  $S^2$ . (The whole of the ellipsoid lies within  $B^3$ , since  $b_1^2 + b_2^2 + b_3^2 \leq b_1^2 + 2b_2^2 + 3b_3^2 = 1/2 < 1$ .)

**Problem 12:** We view  $L_{3/2} = f^{-1}(3/2) \subset \mathbb{R}P^3$  as the quotient space  $\pi(\tilde{L}_{3/2})$  of the level set  $\tilde{L}_{3/2} = g^{-1}(3/2) \subset S^3$ . Here

$$\tilde{L}_{3/2} = \{(a_0, a_1, a_2, a_3) \in S^3 \mid a_1^2 + 2a_2^2 + 3a_3^2 = 3/2\}.$$

We fold  $S^3$  onto the tetrahedron

$$\Delta^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x + y + z \leq 1\}$$

by  $\phi: (a_0, a_1, a_2, a_3) \mapsto (a_1^2, a_2^2, a_3^2)$ . This maps each of the regions

$$\{(a_0, a_1, a_2, a_3) \in S^3 \mid \epsilon_0 a_0 \geq 0, \epsilon_1 a_1 \geq 0, \epsilon_2 a_2 \geq 0, \epsilon_3 a_3 \geq 0\}$$

homeomorphically onto  $\Delta^3$ , where  $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}$  range over the  $2^4 = 16$  possible choices of signs. The folded image of  $\tilde{L}_{3/2}$  is the intersection

$$\phi(\tilde{L}_{3/2}) = \{(x, y, z) \in \Delta^3 \mid x + 2y + 3z = 3/2\}$$

of  $\Delta^3$  with the plane  $x + 2y + 3z = 3/2$ , which equals the quadrangle  $ABCD$  spanned by

$$\begin{aligned} A &= (3/4, 0, 1/4) \\ B &= (1/2, 1/2, 0) \\ C &= (0, 3/4, 0) \\ D &= (0, 0, 1/2). \end{aligned}$$

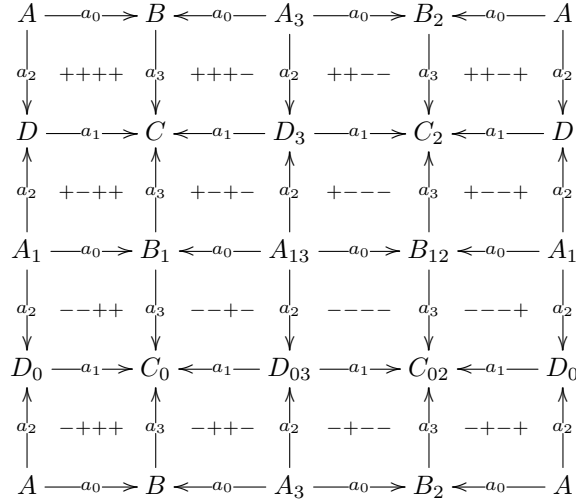
The points with  $a_0 = 0$ ,  $a_1 = 0$ ,  $a_2 = 0$  and  $a_3 = 0$  map onto the edges  $AB$ ,  $CD$ ,  $AD$  and  $BC$ , respectively. Hence

$$\tilde{L}_{3/2} = \phi^{-1}(ABCD)$$

is the union of sixteen topological quadrangles of the form

$$\{(a_0, a_1, a_2, a_3) \in \tilde{L}_{3/2} \mid \epsilon_0 a_0 \geq 0, \epsilon_1 a_1 \geq 0, \epsilon_2 a_2 \geq 0, \epsilon_3 a_3 \geq 0\},$$

glued together as indicated below. Each quadrangle is drawn as a square, with the sequence of signs  $\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3$  in the center. The edges where  $a_i = 0$  are labeled  $a_i$ , and crossing such an edge changes the sign of  $\epsilon_i$ . The subscripts on  $A$ ,  $B$ ,  $C$  or  $D$  show which  $\epsilon_i$  are negative throughout a neighborhood of that point.



The top side is equal to the bottom side, and the left hand side is equal to the right hand side. Hence  $\tilde{L}_{3/2}$  is topologically a torus, homeomorphic to  $S^1 \times S^1$ .

To pass from  $\tilde{L}_{3/2}$  to the quotient  $L_{3/2}$  we identify the quadrangle labeled  $\epsilon_0\epsilon_1\epsilon_2\epsilon_3$  with the one labeled  $\eta_0\eta_1\eta_2\eta_3$ , where each  $\eta_i$  is the opposite sign to  $\epsilon_i$ . Hence the lower half of the diagram is identified with the upper half, by the offset going two squares down and two squares to the right (interpreted cyclically). This leaves the rectangle

$$\begin{array}{ccccccc}
 A & \xrightarrow{a_0} & B & \xleftarrow{a_0} & A & \xrightarrow{a_0} & B & \xleftarrow{a_0} & A \\
 \downarrow a_2 & +++ & \downarrow a_3 & +++ & \downarrow a_2 & ++- & \downarrow a_3 & +-+ & \downarrow a_2 \\
 D & \xrightarrow{a_1} & C & \xleftarrow{a_1} & D & \xrightarrow{a_1} & C & \xleftarrow{a_1} & D \\
 \uparrow a_2 & -++ & \uparrow a_3 & -+- & \uparrow a_2 & --- & \uparrow a_3 & +-- & \uparrow a_2 \\
 A & \xrightarrow{a_0} & B & \xleftarrow{a_0} & A & \xrightarrow{a_0} & B & \xleftarrow{a_0} & A
 \end{array}$$

with the left hand side equal to the right hand side, and the upper side identified with the lower side, after half a rotation. Cutting and pasting along the dotted line shows that this space, homeomorphic to  $L_{3/2}$ , is also a topological torus.

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