

INTEGRATION ON MANIFOLDS

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§23.3: The integral of an n -form on $\mathbb{R}^n$.

Theorem: Let $U, V \subset \mathbb{R}^n$ be connected and open, let $\phi: U \rightarrow V$ be a diffeomorphism, and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous with compact $\text{supp}(f) \subset V$. Then

$$\begin{aligned} \int_V f \, dy^1 \dots dy^n &= \pm \int_U f \phi \cdot \det \text{Jac}(\phi) \, dx^1 \dots dx^n \\ &= \int_U f \phi \cdot |\det \text{Jac}(\phi)| \, dx^1 \dots dx^n, \end{aligned}$$

where $\text{Jac}(\phi) = (\partial y^i \phi / \partial x^j)_{i,j}$ and the sign is $+$ if ϕ preserves the standard orientation and $-$ if it reverses it.

Proof for $n = 1$: Set $U = (a, b)$ and $\phi(U) = V = (c, d)$. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be an antiderivative of f , so that $F'(y) = f(y)$ and

$$(F\phi)'(x) = F'(\phi(x)) \cdot \phi'(x) = (f\phi)(x) \cdot \phi'(x).$$

Then

$$[F\phi]_a^b = \int_a^b (F\phi)'(x) \, dx = \int_a^b f\phi \cdot \phi' \, dx$$

is equal to

$$[F]_{\phi(a)}^{\phi(b)} = \int_{\phi(a)}^{\phi(b)} F'(y) \, dy = \int_{\phi(a)}^{\phi(b)} f \, dy = \pm \int_c^d f \, dy.$$

QED

Definition: If $\omega = f \, dy^1 \wedge \dots \wedge dy^n$ is an n -form with compact $\text{supp}(\omega) = \text{supp}(f) \subset V \subset \mathbb{R}^n$ let

$$\int_V \omega = \int_V f \, dy^1 \wedge \dots \wedge dy^n$$

be equal to

$$\int_V f \, dy^1 \dots dy^n = \int_{\mathbb{R}} (\dots (\int_{\mathbb{R}} f \, dy^1) \dots) \, dy^n.$$

Proposition: Let $\phi: U \rightarrow V$ be orientation-preserving. Then

$$\int_U \phi^* \omega = \int_V \omega.$$

Proof: Note that

$$\phi^* \omega = f \phi \cdot \det \text{Jac}(\phi) \, dx^1 \wedge \dots \wedge dx^n.$$

Date: April 2022.

Hence

$$\begin{aligned}\int_V \omega &= \int_V f dy^1 \wedge \cdots \wedge dy^n \\ &= \int_V f dy^1 \cdots dy^n = \int_U f \phi \cdot \det \text{Jac}(\phi) dx^1 \cdots dx^n \\ &= \int_U f \phi \cdot \det \text{Jac}(\phi) dx^1 \wedge \cdots \wedge dx^n = \int_U \phi^* \omega.\end{aligned}$$

QED

Proposition: Let $\phi: U \rightarrow V$ be orientation-reversing. Then

$$\int_U \phi^* \omega = - \int_V \omega.$$

The proof is similar.

§23.4: Integration of an n -form on an n -manifold.

Definition: Let M be a smooth n -manifold with boundary. For each $k \geq 0$ let

$$\Omega_c^k(M) \subset \Omega^k(M)$$

denote the subspace of k -forms with compact support. If M is compact, then each k -form has compact support.

Suppose now that M is oriented.

Definition: Let $\omega \in \Omega_c^n(M)$ be an n -form with compact support. If $\text{supp}(\omega) \subset V$ lies in a single chart domain, with $(V, y: V \rightarrow \mathbb{H}^n)$ in the oriented atlas for M , define

$$\int_M \omega = \int_V \omega$$

to be

$$\int_{y(V)} (y^{-1})^* \omega = \int_{\mathbb{R}^n} (y^{-1})^* \omega.$$

Here $y^{-1}: y(V) \rightarrow V$ and $(y^{-1})^*(\omega) \in \Omega_c^n(y(V))$ where $y(V) \subset \mathbb{H}^n \subset \mathbb{R}^n$.

More explicitly, if $\omega = f dy^1 \wedge \cdots \wedge dy^n$ where f has compact $\text{supp}(f) \subset V$, then

$$(y^{-1})^* \omega = f y^{-1} dr^1 \wedge \cdots \wedge dr^n$$

in $\Omega_c^n(y(V))$, so

$$\int_V \omega = \int_{\mathbb{R}^n} (y^{-1})^* \omega = \int_{\mathbb{R}^n} f y^{-1} dr^1 \cdots dr^n.$$

Lemma: If $\text{supp}(\omega) \subset U$ also lies in another chart domain, with $(U, x: U \rightarrow \mathbb{H}^n)$ in the oriented atlas for M , then

$$\int_U \omega = \int_{x(U)} (x^{-1})^* \omega$$

is equal to

$$\int_V \omega = \int_{y(V)} (y^{-1})^* \omega.$$

Hence $\int_M \omega$ is well-defined.

Proof: We may replace both U and V by $U \cap V$, and hence assume that $U = V$. Let $\phi = yx^{-1}: x(U) \rightarrow y(V)$. Then $x^{-1} = y^{-1}\phi$, so $(x^{-1})^* \omega = \phi^*(y^{-1})^* \omega$ and

$$\int_{x(U)} (x^{-1})^* \omega = \int_{x(U)} \phi^*(y^{-1})^* \omega = \int_{y(V)} (y^{-1})^* \omega,$$

since ϕ is an orientation-preserving diffeomorphism. QED

We use partitions of unity to extend to the global case. Let $\omega \in \Omega_c^n(M)$ have compact support $\text{supp}(\omega) \subset M$. Choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to an oriented atlas $\{(U_\alpha, x_\alpha)\}_\alpha$ for M . Then

$$\omega = \sum_{\alpha} \rho_{\alpha} \omega$$

has only finitely many nonzero terms. This holds locally, since $\{\rho_\alpha\}_\alpha$ is locally finite, hence also globally, since $\text{supp}(\omega)$ is compact. We can therefore treat this as a finite sum. Each $\text{supp}(\rho_\alpha \omega) \subset U_\alpha$ is compact, so the integral

$$\int_{U_\alpha} \rho_\alpha \omega$$

is defined in $\mathbb{R}$.

Definition: Let

$$\int_M \omega = \sum_{\alpha} \int_{U_\alpha} \rho_\alpha \omega.$$

Lemma: If $\{(V_\beta, y_\beta)\}_\beta$ is a second oriented atlas for (the same orientation on) M , and $\{\chi_\beta\}_\beta$ is a partition of unity subordinate to this atlas, then

$$\sum_{\alpha} \int_{U_\alpha} \rho_\alpha \omega = \sum_{\beta} \int_{V_\beta} \chi_\beta \omega,$$

so $\int_M \omega$ is well-defined.

Proof:

$$\sum_{\alpha} \int_{U_\alpha} \rho_\alpha \omega = \sum_{\alpha} \int_{U_\alpha} \rho_\alpha \sum_{\beta} \chi_\beta \omega = \sum_{\alpha} \sum_{\beta} \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega$$

since $\sum_{\beta} \chi_\beta = 1$ and $\text{supp}(\rho_\alpha \chi_\beta \omega) \subset U_\alpha \cap V_\beta$. This is symmetric in α/β , U/V and ρ/χ . QED

Integration over an oriented n -manifold M defines a linear operator

$$\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}.$$

If M is compact, this is defined on $\Omega^n(M)$.

Proposition 23.9: Let $-M$ denote M with the opposite orientation. Then

$$\int_{-M} \omega = - \int_M \omega.$$

Proof: It suffices to show this for $\text{supp}(\omega) \subset V$ with (V, y) in the oriented atlas for M . Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an orientation-reversing diffeomorphism, and let $x = \phi^{-1}y$, so that (V, x) is in the oriented atlas for $-M$. Then $(x^{-1})^* = \phi^*(y^{-1})^*$, so

$$\begin{aligned} \int_{-M} \omega &= \int_{-V} \omega = \int_{x(V)} (x^{-1})^* \omega = \int_{x(V)} \phi^*(y^{-1})^* \omega \\ &= - \int_{y(V)} (y^{-1})^* \omega = - \int_V \omega = - \int_M \omega. \end{aligned}$$

QED

When $n = 0$, so that $M = \{p_i\}_i \cup \{q_j\}_j$ where each p_i has orientation $+1$ and each q_j has orientation -1 , we define

$$\int_M f = \sum_i f(p_i) - \sum_j f(q_j)$$

for $f: M \rightarrow \mathbb{R}$ with compact support.

§23.5: Stokes' theorem.

Theorem 23.12: Let M be an oriented n -manifold with boundary ∂M , equipped with the boundary orientation. Let ω be a smooth $(n-1)$ -form with compact support on M , and also write ω for its restriction to ∂M . Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof: First prove this for $M = \mathbb{R}^n$ and for $M = \mathbb{H}^n$. We do the case of $M = \mathbb{H}^3$, with coordinates (x, y, z) . The standard orientation of M is detected by $dx \wedge dy \wedge dz$, while the boundary orientation of ∂M is detected by

$$\iota_{-\partial/\partial z}(dx \wedge dy \wedge dz) = -dx \wedge dy,$$

i.e., the opposite of the standard orientation on $\mathbb{R}^2$. The 2-form ω is a linear combination

$$\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

for smooth and compactly supported functions $f, g, h: \mathbb{H}^3 \rightarrow \mathbb{R}$. By definition,

$$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz,$$

while

$$\omega(x, y) = h(x, y, 0) dx \wedge dy$$

when restricted to ∂M . Choose $a > 0$ so large that each support lies within $[-a, a] \times [-a, a] \times [0, a]$. Then

$$\begin{aligned} \int_M d\omega &= \int_{\mathbb{H}^3} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx dy dz = \iint_{[-a, a] \times [0, a]} \left(\int_{-a}^a \frac{\partial f}{\partial x} dx\right) dy dz \\ &\quad + \iint_{[-a, a] \times [0, a]} \left(\int_{-a}^a \frac{\partial g}{\partial y} dy\right) dx dz + \iint_{[-a, a] \times [-a, a]} \left(\int_0^a \frac{\partial h}{\partial z} dz\right) dx dy \\ &= 0 + 0 + \iint_{[-a, a] \times [-a, a]} (0 - h(x, y, 0)) dx dy \\ &= - \int_{\mathbb{R}^2} h(x, y, 0) dx dy = \int_{\partial M} \omega. \end{aligned}$$

For a general M , choose an atlas $\{(U_\alpha, x_\alpha)\}_\alpha$ where each U_α is oriented diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$ with the standard orientation. The theorem then holds for each U_α , with $\partial U_\alpha = \partial M \cap U_\alpha$.

Choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to $\{U_\alpha\}_\alpha$. Then

$$\begin{aligned}\int_{\partial M} \omega &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega \\ &= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) \\ &= \sum_\alpha \int_M d(\rho_\alpha \omega) = \int_M d(\sum_\alpha \rho_\alpha \omega) = \int_M d\omega.\end{aligned}$$

QED

§23.6: Line integrals and Green's theorem.

Let $C \subset \mathbb{R}^n$ be an oriented smooth curve with $\partial C = \{+p, -q\}$. Let $f: C \rightarrow \mathbb{R}$ be a smooth function with compact support. Then the line integral

$$\int_C df = \int_C \sum_{i=1}^n \frac{\partial f}{\partial r^i} dr^i = \int_C \text{grad } f \cdot dr$$

is equal to

$$\int_{\partial C} f = f(p) - f(q).$$

Let $S \subset \mathbb{R}^3$ be an oriented smooth surface with boundary ∂S . Let

$$X = (P, Q, R)$$

be a smooth vector field on S with compact support. Let

$$\omega = P dx + Q dy + R dz$$

so that

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

with

$$\text{curl } X = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Then the surface integral

$$\int_S d\omega = \iint_S \text{curl } X \cdot dS$$

is equal to the line integral

$$\int_{\partial S} \omega = \oint_{\partial S} X \cdot dr.$$

Let $M \subset \mathbb{R}^n$ be an n -dimensional regular submanifold with boundary ∂M and the standard orientation, and let

$$X = (P_1, \dots, P_n) = \sum_{i=1}^n P_i \frac{\partial}{\partial r^i}$$

be a smooth vector field on M with compact support. Let

$$\omega = \iota_X(dr^1 \wedge \dots \wedge dr^n) = \sum_{i=1}^n (-1)^{i-1} P_i dr^1 \wedge \dots \wedge \widehat{dr^i} \wedge \dots \wedge dr^n$$

so that

$$d\omega = \sum_{i=1}^n \frac{\partial P_i}{\partial r^i} dr^1 \wedge \cdots \wedge dr^n$$

with

$$\operatorname{div} X = \sum_{i=1}^n \frac{\partial P_i}{\partial r^i}.$$

Then

$$\int_M d\omega = \int_M \operatorname{div} X \, d\operatorname{Vol}$$

is equal to

$$\int_{\partial M} \omega = \int_{\partial M} X \cdot dN$$

where N is the outward unit normal vector field.