

MAT4510, Fall 2022

Solutions to assignment

Problem 1. Let

$$L = \{(x, y) \in \mathbb{R}^2 : x^3 = y^5\}.$$

Show that $L - \{0\}$ is a (regular) submanifold of $\mathbb{R}^2$, but that L is not.

Solution: Let

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto x^3 - y^5.$$

Then $L = f^{-1}(0)$. Because

$$\frac{\partial f}{\partial x} = 3x^2, \quad \frac{\partial f}{\partial y} = -5y^4,$$

the origin is the only critical point of f . Therefore, $L - \{0\}$ is a submanifold of $\mathbb{R}^2$. Now suppose L were a submanifold of $\mathbb{R}^2$. Then there would exist an immersion

$$\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^2$$

taking values on L and satisfying $\gamma(0) = 0$. Writing α, β for the components of γ we then have

$$\alpha(t)^3 = \beta(t)^5$$

for all t . Because $\gamma'(0) \neq 0$, either $\alpha'(0) \neq 0$ or $\beta'(0) \neq 0$.

Case 1. Suppose $\alpha'(0) \neq 0$. Then α is invertible in a neighbourhood of the origin, hence there is a smooth function

$$u : (-\delta, \delta) \rightarrow \mathbb{R}$$

such that $u(0) = 0$ and $\alpha \circ u = \text{Id}$. Then for $s \in (-\delta, \delta)$ one has

$$s^3 = \alpha(u(s))^3 = \beta(u(s))^5.$$

But then

$$\beta(u(s)) = \sqrt[5]{s^3},$$

which is not smooth, a contradiction.

Case 2. If $\beta'(0) \neq 0$ one obtains a contradiction in the same way as in Case 1.

Problem 2.

(i) Show that the map

$$F : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2 - \{0\}, \quad (x, y) \mapsto \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

is a diffeomorphism.

Solution: Since both components of F are rational functions, the map F is smooth. It is easily verified that $F \circ F = \text{Id}$, hence F is bijective and $F^{-1} = F$ is smooth. Therefore, F is a diffeomorphism.

(ii) Let V, W be the vector fields on $\mathbb{R}^2 - \{0\}$ given by

$$V = \frac{\partial}{\partial x}, \quad W = (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y}.$$

Show that V is F -related to W .

Solution: For clarity of notation we write $\partial_i f$ for the i th partial derivative of a function f , so that $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$. Let F^1, F^2 be the components of F . Then

$$\begin{aligned} F_*(V_{(x,y)}) &= \partial_1 F^1(x, y) \cdot \partial_1|_{F(x,y)} + \partial_1 F^2(x, y) \cdot \partial_2|_{F(x,y)} \\ &= (x^2 + y^2)^{-2} ((y^2 - x^2) \partial_1|_{F(x,y)} - 2xy \partial_2|_{F(x,y)}) \\ &= W_{F(x,y)}, \end{aligned}$$

showing that V is F -related to W .

(iii) Let

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$$

be the unit sphere, and let $N = (0, 0, 1)$ be the north pole in S^2 . Let

$$\tau_+ : S^2 - \{N\} \rightarrow \mathbb{R}^2$$

be stereographic projection as defined in Problem sheet 2. Show that there exists a smooth vector field Z on S^2 whose restriction to $S^2 - \{N\}$ is τ_+ -related to V . How many zeros does Z have?

Solution: Let $\tau_- : S^2 - \{-N\} \rightarrow \mathbb{R}^2$ be stereographic projection from the south pole. We now regard V, W as vector fields on all of $\mathbb{R}^2$ given by the

same formulas as in (ii). We define smooth vector fields $Z_{\pm}$ on $S^2 - \{\pm N\}$ by

$$Z_+ = (\tau_+)^*V, \quad Z_- = (\tau_-)^*W.$$

By the result of Sheet 2, Problem 2(ii), we have that $F = \tau_- \circ \tau_+^{-1}$. For $p \in S^2 - \{\pm N\}$ and $q = \tau_+(p)$ we therefore have

$$W_{\tau_-(p)} = F_*V_q = (\tau_-)_*(\tau_+^{-1})_*V_q = (\tau_-)_*(Z_+)_p.$$

This shows that $Z_+ = Z_-$ on $S^2 - \{\pm N\}$, so $Z_{\pm}$ patch together to a smooth vector field Z on all of S^2 . Because V has no zeros, it follows that Z does not vanish on $S^2 - \{N\}$. On the other hand, W vanishes at the origin, so Z vanishes at $N = \tau_-^{-1}(0)$. Hence, Z has exactly one zero.

Problem 3.

(i) Show that the map

$$F : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{C}^2, \quad (t, z) \mapsto (z^2, tz).$$

restricts to an immersion $f : S^2 \rightarrow \mathbb{C}^2$, where S^2 is now regarded as the unit sphere in $\mathbb{R} \times \mathbb{C}$, i.e.

$$S^2 = \{(t, z) \in \mathbb{R} \times \mathbb{C} : t^2 + |z|^2 = 1\}.$$

Solution: The derivative of F at a point (t, z) is given by

$$DF_{(t,z)}(s, w) = (2zw, sz + tw).$$

If $z \neq 0$ then $DF_{(t,z)}$ is clearly injective, whereas if $z = 0$ but $t \neq 0$ then the kernel of $DF_{(t,z)}$ equals $\mathbb{R} \times \{0\}$. On the other hand, for $(t, z) \in S^2$, the tangent space $T_{(t,z)}S^2$, regarded as a subspace of $\mathbb{R} \times \mathbb{C}$, is

$$T_{(t,z)}S^2 = \{(s, w) \in \mathbb{R} \times \mathbb{C} : st + w\bar{z} = 0\}.$$

It follows from this that $DF_{(t,z)}$ restricts to an injective map $T_{(t,z)}S^2 \rightarrow \mathbb{C}^2$, hence F restricts to an immersion $f : S^2 \rightarrow \mathbb{C}^2$.

(ii) Let $\pi : S^2 \rightarrow \mathbb{RP}^2$ be the projection. Show that there exists a unique map $g : \mathbb{RP}^2 \rightarrow \mathbb{C}^2$ such that

$$f = g \circ \pi,$$

and that g is a smooth embedding.

Solution: The projection π satisfies

$$\pi(t, z) = \pi(s, w) \iff (t, z) = \pm(s, w).$$

Because $f(t, z) = f(-t, -z)$ there is a map $g : \mathbb{RP}^2 \rightarrow \mathbb{C}^2$ such that $f = g \circ \pi$. Since π is surjective, the map g is unique.

We now show that g is injective. Suppose $f(t, z) = f(s, w)$, i.e.

$$z^2 = w^2, \quad tz = sw.$$

We will prove that

$$(t, z) = \pm(s, w), \tag{1}$$

so that $\pi(t, z) = \pi(s, w)$. If $z \neq 0$ then $z = \epsilon w$ and $t = \epsilon s$ for some $\epsilon = \pm 1$, from which (1) follows. If instead $z = 0$ then (t, z) and (s, w) both have the form $(\pm 1, 0)$, so (1) holds in this case as well.

Because π is a local diffeomorphism (by Sheet 3, Problem 3) and f is an immersion, it follows that g is also an immersion. In particular, g is continuous. Since $\mathbb{RP}^2$ is compact and $\mathbb{C}^2$ is Hausdorff, the map g is a topological embedding. Altogether, this shows that g is a smooth embedding.

Problem 4. Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}.$$

(i) Show that G is a submanifold of the space $M(2 \times 2; \mathbb{R})$ of 2×2 real matrices.

Solution: Let U be the open subset of $M := M(2 \times 2; \mathbb{R})$ consisting of all matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x > 0$, and let

$$\phi : U \rightarrow \mathbb{R}^4, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto (x, y, z, w - 1).$$

Then ϕ is an open embedding, and $U \cap G$ is the locus in U where the last two components of ϕ vanish. Thus, (U, ϕ) is a chart on M adapted to G . Because $G \subset U$, this proves that G is a submanifold of M .

(ii) Show that G is a Lie group.

Solution: We compute

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xx' & xy' + y \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x^{-1} & -x^{-1}y \\ 0 & 1 \end{pmatrix},$$

hence G is a subgroup of $\mathrm{GL}(2 \times 2, \mathbb{R})$. By (i), G is also a submanifold of $\mathrm{GL}(2 \times 2, \mathbb{R})$ and therefore a Lie group.

(iii) Find a basis A, B for the Lie algebra $\mathfrak{g}$ of G , regarded as a subspace of $M(2 \times 2; \mathbb{R})$, and compute $[A, B]$.

Solution: The Lie algebra of G has the basis

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have $AB = B$ and $BA = 0$, so

$$[A, B] = AB - BA = B.$$