

MAT4520

Mandatory assignment 1 of 1

Submission deadline

Thursday 27nd April 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with L^AT_EX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

In order to pass the assignment you need a score of at least 66%.

Problem 1. Let

$$L = \{(x, y) \in \mathbb{R}^2 : x^3 = y^5\}.$$

Show that $L - \{0\}$ is a (regular) submanifold of $\mathbb{R}^2$, but that L is not.

Problem 2.

(i) Show that the map

$$F : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2 - \{0\}, \quad (x, y) \mapsto \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

is a diffeomorphism.

(ii) Let V, W be the vector fields on $\mathbb{R}^2 - \{0\}$ given by

$$V = \frac{\partial}{\partial x}, \quad W = (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y}.$$

Show that V is F -related to W .

(iii) Let

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$$

be the unit sphere, and let $N = (0, 0, 1)$ be the north pole in S^2 . Let

$$\tau_+ : S^2 - \{N\} \rightarrow \mathbb{R}^2$$

be stereographic projection as defined in Problem sheet 2. Show that there exists a smooth vector field Z on S^2 whose restriction to $S^2 - \{N\}$ is τ_+ -related to V . How many zeros does Z have?

Continued on the next page!

Problem 3.

(i) Show that the map

$$F : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{C}^2, \quad (t, z) \mapsto (z^2, tz).$$

restricts to an immersion $f : S^2 \rightarrow \mathbb{C}^2$, where S^2 is now regarded as the unit sphere in $\mathbb{R} \times \mathbb{C}$, i.e.

$$S^2 = \{(t, z) \in \mathbb{R} \times \mathbb{C} : t^2 + |z|^2 = 1\}.$$

(ii) Let $\pi : S^2 \rightarrow \mathbb{RP}^2$ be the projection. Show that there exists a unique map $g : \mathbb{RP}^2 \rightarrow \mathbb{C}^2$ such that

$$f = g \circ \pi,$$

and that g is a smooth embedding.

Problem 4. Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}.$$

(i) Show that G is a submanifold of the space $M(2 \times 2; \mathbb{R})$ of 2×2 real matrices.

(ii) Show that G is a Lie group.

(iii) Find a basis A, B for the Lie algebra $\mathfrak{g}$ of G , regarded as a subspace of $M(2 \times 2; \mathbb{R})$, and compute $[A, B]$.