

MAT4520 - SELECTED TOPICS FOR REPETITION, SUGGESTED BY THE STUDENTS

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1. TANGENT SPACES AND TANGENT BUNDLES

1.1. The tangent space at a point. Let M be a smooth manifold and let $p \in M$. A *tangent vector* at p is a linear map $D : C_p^\infty \rightarrow \mathbb{R}$ (where C_p^∞ denotes the space of germs of smooth functions at p) such that the Leibniz rule is satisfied:

$$D(fg) = Df \cdot g(p) + f(p) \cdot Dg$$

for all $f, g \in C_p^\infty$. Such a map D is also called a *derivation* at p . The set of derivations at p has the natural structure of an $\mathbb{R}$ -vector space, and we call this space the *tangent space* of M at p , denoting it by $T_p M$.

1.2. The tangent space at a point $p \in \mathbb{R}^n$. Let $D \in T_p \mathbb{R}^n$ and let f be a smooth function at p . By Taylor's Theorem we have that

$$f(x) = f(p) + \sum_{j=1}^n (x^j - p^j) \cdot g_j(x)$$

where the g_j 's are smooth at p , and $g_j(p) = \frac{\partial f}{\partial x^j}(p)$. We get that

$$\begin{aligned} Df &= D(f(p)) + \sum_{j=1}^n D((x^j - p^j))g_j(p) + (x^j - p^j)(p)Dg_j \\ &= \sum_{j=1}^n Dx^j \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

where we have used that for a constant $c \in \mathbb{R}$, we have that $Dc = 0$. Noting that each partial derivative $\frac{\partial}{\partial x^j}|_p$ is a derivation at p , we see that the set

$$\left\{ \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p \right\}$$

spans $T_p \mathbb{R}^n$. But it is easy to see that these derivations are also linearly independent since

$$\left(\sum_{j=1}^n v_j \frac{\partial}{\partial x^j}|_p \right)(x^i) = v^i$$

so this set is actually a basis for $T_p \mathbb{R}^n$.

For each point $p \in \mathbb{R}^n$ we may now identify the space $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself via the identification

$$D = \sum_{j=1}^n v_j \frac{\partial}{\partial x^j} \Big|_p \leftrightarrow (v_1, \dots, v_n) = v$$

Then D is the directional derivative in the direction v at p .

For any open set $U \subset \mathbb{R}^n$ we may repeat the discussion above and define the tangent space $T_p U$ for each $p \in U$; there will be no difference between $T_p U$ and $T_p \mathbb{R}^n$.

1.3. The tangent bundle of an open set $U \subset \mathbb{R}^n$. The tangent bundle TU of U is initially the disjoint union of the spaces $T_p U$. We have that TU is naturally identified with $U \times \mathbb{R}^n$ via the identification

$$\sum_{j=1}^n v_j \frac{\partial}{\partial x^j} \Big|_p \leftrightarrow (p, v)$$

This gives TU a topology, demanding that this identification is a homeomorphism.

1.4. The Chain Rule. Suppose that $F : M \rightarrow N$ is a smooth map between manifolds. Then for $p \in M$ there is an induced linear map

$$F_* : T_p M \rightarrow T_{F(p)} N$$

defined for $X_p \in T_p M$ and $f \in C_{F(p)}^\infty$ by

$$F_* X_p(f) := X_p(f \circ F)$$

Now if $G : N \rightarrow P$ is also a smooth map between manifolds we can consider the composition $G_{*,F(p)} \circ F_{*,p}$. By definition, for $X_p \in T_p M$ and $f \in C_{G(F(p))}^\infty$ we have that

$$G_*(F_* X_p)(f) = F_* X_p(f \circ G) = X_p(f \circ G \circ F) = (G \circ F)_* X_p(f)$$

so we see that $(G \circ F)_* = G_* \circ F_*$.

It follows from this that if F were a local diffeomorphism near p then $F_{*,p}$ is actually an isomorphism of vector spaces. Letting G denote the local inverse to F at $q = F(p)$ we have that

$$\text{Id} = (G \circ F)_{*,p} = G_{*,q} \circ F_{*,p}$$

and similarly $\text{Id} = (F \circ G)_{*,q} = F_{*,p} \circ G_{*,q}$.

1.5. Bases for The Tangent Space at a point $p \in M$ for a smooth manifold M . Let $(U, \phi = (x^1, \dots, x^n))$ be a chart on an n -manifold M . Then

$$\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$$

is a diffeomorphism, and so

$$\phi_{*,p} : T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n$$

is an isomorphism. Letting $r^1, \dots, r^n$ denote the standard coordinates on $\mathbb{R}^n$ we have seen that

$$\left\{ \frac{\partial}{\partial r^1} |_q, \dots, \frac{\partial}{\partial r^n} |_q \right\}$$

is a basis for $T_q \mathbb{R}^n$, and so the set

$$\{(\phi^{-1})_{*,q} \frac{\partial}{\partial r^j} |_q\}_{j=1}^n =: \left\{ \frac{\partial}{\partial x^j} |_p \right\}_{j=1}^n$$

is a basis for $T_p M$. In particular we have that $T_p M$ has dimension n .

Assume that we have an additional chart $(V, \psi = y^1, \dots, y^n)$ near the point p . Then any tangent vector $v \in T_p$ has vector representations $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ with respect to the corresponding two bases. We would like to give an expression for a transition matrix transforming a to b . For this we write

$$\frac{\partial}{\partial x^j} = \sum_k c_j^k \frac{\partial}{\partial y^k}$$

Applying both sides to the coordinate function y^i we see that $c_j^i = \frac{\partial y^i}{\partial x^j}$ (using that $\frac{\partial y^i}{\partial y^k} = \delta_k^i$). From this it follows that

$$b = \left[\frac{\partial y^i}{\partial x^j} (p) \right] a$$

1.6. The Tangent Bundle of a manifold M . As a set the *tangent bundle* TM of a smooth manifold M is formed by considering the disjoint union of the tangent spaces $T_p M$. There is a natural map $\pi : TM \rightarrow M$ defined by $\pi(X_p) = p$ for $X_p \in T_p M$.

We would like to turn TM into a smooth manifold. For each chart (U, ϕ) on M there is a 1 – 1 correspondence $\tilde{\phi}$ between TU and $T\phi(U)$ by setting $\tilde{\phi}(X_p) := \phi_{*,p} X_p$ for $X_p \in T_p M$. Then

$$\tilde{\phi}(X_p) = \tilde{\phi} \left(\sum_j a_j \frac{\partial}{\partial x^j} |_p \right) = \sum_j a_j \frac{\partial}{\partial r^j} |_{\phi(p)}$$

Using the identification of $T\phi(U)$ with $\phi(U) \times \mathbb{R}^n$ as above, we get that

$$\tilde{\phi}(X_p) = (\phi(p), a)$$

We give TU a topology by declaring that $\tilde{\phi}$ is a homeomorphism. Doing this for each chart (U, ϕ) it turns out that TM is given the structure of a locally Euclidean topological space, which is Hausdorff and second countable.

We will finally see that with the collection of charts $(TU, \tilde{\phi})$ we have that TM is given the structure of a smooth manifold, i.e., we will see that the charts are smoothly compatible. So let $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ be two such charts, with $U \cap V \neq \emptyset$. Letting (r, a) denote the coordinates on $T\phi(U)$ and using the transition matrix from the previous subsection we have that

$$\tilde{\psi} \circ (\tilde{\phi})^{-1}(r, a) = (\psi \circ \phi^{-1}(r), J(\psi \circ \phi^{-1})(r)a)$$

on $\tilde{\phi}(T(U \cap V))$ which is indeed smooth.

2. BUMP FUNCTIONS AND PARTITIONS OF UNITY

2.1. Bump functions. Let M be a manifold, let $q \in M$, and let V be an open set containing q . A *bump function at q supported in V* is a nonnegative continuous function ρ on M such that $\rho = 1$ near q and $\text{Supp } \rho \subset V$. We will be interested in smooth bump functions.

Example 2.1. A smooth nonnegative function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ such that $\rho(x) = 1$ for $|t| \leq a$ and $\rho = 0$ for all $|t| \geq b$, for some $0 < a < b$.

With such a function at hand we can construct bump functions in higher dimensions and on manifolds.

- $\kappa(x) = \rho(\|x\|)$ will be a bump function at zero in $\mathbb{R}^n$ with support in $B(0, b + \epsilon)$ for any $\epsilon > 0$.
- $\tau(x) = \kappa(x - q)$ will be a bump function at q with support in $B(q, b + \epsilon)$ for any $\epsilon > 0$.
- For a point q in a manifold M and an open set V containing q we may choose chart (U, ϕ) centred at q , and for sufficiently small b we have that $\kappa \circ \phi$ is a bump function at q supported in V .

We have seen several applications of bump functions throughout the course: globalizing smooth functions, vector fields, local coordinates etc. They are also used to construct partitions of unity, which we have used to define integration on manifolds. See also the example given below.

A bump function as in the example is constructed in several steps.

Step 1: Consider $f(t)$ defined by $f(t) = 0$ for $t \leq 0$ and $f(t) = e^{-1/t}$ for $t > 0$. The exponential function e^x decays so fast when $x \rightarrow -\infty$, that f is smooth.

Step 2: Set $g(t) = \frac{f(t)}{f(t) + f(1-t)}$. Then g is smooth, and $g(t) = 0$ for $t \leq 0$, $g(t) = 1$ for $t \geq 1$.

Step 3: Set $h(t) = g(\frac{t-a}{b-a})$.

Step 4: Set $k(t) = h(t^2)$.

Step 5: Set $\rho(t) = 1 - k(t)$.

Remark 2.2. The intermediate functions can also be useful. For instance in the next example when we glue isotopies.

Example 2.3. Suppose that $f, g, h : M \rightarrow N$ are smooth maps between manifolds, and that $f \sim g, g \sim h$. We want to show that $f \sim h$. Let $F : M \times \mathbb{R} \rightarrow N$ be smooth such that $F(\cdot, 0) = f, F(\cdot, 1) = g$ and let $G : M \times \mathbb{R} \rightarrow N$ be smooth such that $G(\cdot, 0) = g, G(\cdot, 1) = h$.

An initial try could be to define $H : M \times \mathbb{R} \rightarrow N$ by $H = F$ for $t \leq 1$, $H(\cdot, t) = G(\cdot, t - 1)$ for $t \geq 1$, and then $\tilde{H}(x, t) = H(x, 2t)$. The problem with this is that there is no reason we this map should be smooth at $t = 1/2$.

The trick is to modify the initial F and G . Letting h be a function as above with $a^2 = 1/4$ and $b^2 = 3/4$ we consider the isotopies $\tilde{F}(\cdot, t) = F(\cdot, h(t))$ and $\tilde{G}(\cdot, t) = G(\cdot, h(t))$. Using these new isotopies the initial try will work.

2.2. Partitions of unity. A *smooth partition of unity* on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \rightarrow \mathbb{R}\}_{\alpha \in A}$ such that

- (i) The collection of supports $\{\text{Supp} \rho_\alpha\}$ is locally finite,
- (ii) $\sum \rho_\alpha = 1$

If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M and $\text{Supp} \rho_\alpha \subset U_\alpha$ for all α , we say that the partition of unity is subordinate to the cover.

We will not repeat the proof of the existence of partitions of unities subordinate to any give cover here, but we recall how they were crucial to define integration of differentiable forms on manifolds.

Let M be an oriented manifold, and let ω be a smooth n -form with compact support. Choose an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover. Since the support of ω is compact there are finitely many open sets U_i covering it. Then $\rho_i \omega$ has compact support in U_i and we may define

$$\int_M \omega = \sum_i \int_{U_i} \rho_i \omega$$

Recall that

$$\int_{U_i} \rho_i \omega := \int_{\phi(U_i)} (\phi_i^{-1})^* \rho_i \omega$$

We will show that this definition is independent of the choice of oriented atlas and partition of unity. So let (U_j, ψ_j) be another cover.

$$\begin{aligned}
\sum_i \int_{U_i} \rho_i \omega &= \sum_i \int_{U_i} \rho_i \sum_j \chi_j \omega \\
&= \sum_i \sum_j \int_{U_i} \rho_i \chi_j \omega \\
&= \sum_i \sum_j \int_{U_i \cap V_j} \rho_i \chi_j \omega
\end{aligned}$$

where on $U_i \cap V_j$ have used the chart map ϕ_i to define the integral. However, we have seen that get the same integral if we use ψ_j to define it over $U_i \cap U_j$. By symmetry we get that

$$\sum_i \int_{U_i} \rho_i \omega = \sum_j \int_{V_j} \chi_j \omega$$

3. ORIENTATIONS

3.1. Orientations and integration. The important thing for us regarding orientations on manifolds has been to be able to define integration of n -forms on a manifold M . And the property we really needed was the existence on M of an oriented atlas, i.e., an atlas (U_α, ϕ_α) with the property that for any $U_\alpha \cap U_\beta \neq \emptyset$ we have that

$$\det(J(\psi_\beta \phi_\alpha^{-1})) > 0$$

The reason is the following. For a chart (U, ϕ) and a form ω with compact support in U we defined

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^* \omega$$

If ψ were another chart map on U we write

$$(\psi^{-1})^* \omega = f(r) dy^1 \wedge \cdots \wedge dy^n$$

on $\psi(U)$. Now $\psi \circ \phi^{-1} : \phi(U) \rightarrow \psi(U)$ is a diffeomorphism and we have that

$$(\phi^{-1})^* \omega = (\psi \circ \phi^{-1})^* (\psi^{-1})^* \omega = f(\psi \circ \phi^{-1}) \det(J(\psi_\beta \phi_\alpha^{-1})) dx^1 \wedge \cdots \wedge dx^n$$

and it follows by the change of variable formula that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\psi(U)} (\psi^{-1})^* \omega$$

If we proceed differently (but equivalently) from the book we may define:

A smooth n -manifold is orientable if it has an oriented atlas.

3.2. Relation to standard definition of orientation of vector spaces.

Recall that an orientation on a finite dimensional vector space V is a equivalence class of ordered bases $u = (u_1, \dots, u_n)$ for V . We say that $u \sim v$ if when we form the matrix $[g_j^i]$ determined by

$$u_j = \sum_i g_j^i v_i$$

we have that $\det[g_j^i] > 0$. There are exactly two orientations on V .

For an orientable manifold (as defined above) there is a natural way to assign to each tangent space an orientation. Choose an oriented atlas $(U, \phi = (x^1, \dots, x^n))$ and for each $q \in U$ we let the orientation on $T_q M$ be represented by

$$[(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q)]$$

On overlapping charts we have seen that

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

and so this is globally well defined.

On the other hand, assume that there is an atlas $\{(U_\alpha, \phi_\alpha)\}$ on M and on each U_α continuous vector fields $u^\alpha = (u_1^\alpha, \dots, u_n^\alpha)$ on U_α , such that on each intersection $U_\alpha \cap U_\beta$ the orientations u^α, u^β coincide. Then on each U_α we may alter ϕ_α such that

$$(\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n})$$

determines the same orientation as u^α (this uses continuity). Now, on $U_\alpha \cap U_\beta$ we have that

$$(\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n}) \text{ and } (\frac{\partial}{\partial x_\beta^1}, \dots, \frac{\partial}{\partial x_\beta^n})$$

are equivalent. But we have that

$$\frac{\partial}{\partial x_\alpha^j} = \sum_i \frac{\partial x_\beta^i}{\partial x_\alpha^j} \frac{\partial}{\partial x_\beta^i}$$

and so $\det[\frac{\partial x_\beta^i}{\partial x_\alpha^j}] > 0$, so we have constructed an oriented atlas.

We sum this up with the statement:

A manifold M is orientable if and only if it is continuously orientable.

3.3. Orientation and no-where vanishing top-forms. Suppose now that M is an n -manifold and that there is a smooth n -form on M which is never zero. Then we may choose an atlas (U_α, ϕ_α) such that for any chart (U, α) we have that

$$\omega(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) > 0.$$

We have seen (Lemma 21.1) that if we write

$$\frac{\partial}{\partial x^j} = \sum_i g_j^i \frac{\partial}{\partial y^i}$$

then

$$\omega(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) = \det[g_j^i] \omega(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$$

and since both sides of the equality is positive we have that $\det[g_j^i] > 0$. We have found an oriented atlas.

On the other han, suppose that M is orientable, i.e., M has an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$. Then for any chart $(U, \phi = (x^1, \dots, x^n))$ we have that

$$dx^1 \wedge \dots \wedge dx^n(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) = 1$$

on U . By an argument as above, for any other chart $(V, \psi = (y^1, \dots, y^n))$ with $U \cap V \neq \emptyset$ we have that

$$dx^1 \wedge \dots \wedge dx^n(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}) > 0$$

using that that the atlas is oriented.

Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$, and define

$$\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n$$

Then ω is never zero. For if $p \in M$ there is a chart (U_β, ϕ_β) such that $\rho_\beta(p) > 0$. Then we have that

$$\rho_\beta dx_\beta^1 \wedge \dots \wedge dx_\beta^n(\frac{\partial}{\partial x_\beta^1}, \dots, \frac{\partial}{\partial x_\beta^n}) > 0$$

Further, for all $\alpha \neq \beta$ we have that

$$\rho_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n(\frac{\partial}{\partial x_\beta^1}, \dots, \frac{\partial}{\partial x_\beta^n}) \geq 0$$

and so ω is not zero at p . Since p is arbitrary ω is never zero.

4. DE RHAM COHOMOLOGY

4.1. Definition and some examples. Recall that for a smooth manifold M we have the *de Rham complex*

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

where $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$, $k \geq 0$, is the unique linear map determined by

- (i) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$
- (ii) $d^2 = 0$
- (iii) $df(X) = Xf$ for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$

In local coordinates any $\omega \in \Omega^k(M)$ may be expressed

$$\omega = \sum_I f_I dx^I, dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}, 1 \leq i_1 < i_2 < \dots < i_k \leq n$$

and

$$d\omega = \sum_I df_I \wedge dx^I$$

For $k \in \mathbb{N}$ we define the vector subspaces

$$Z^k(M) = \text{Ker}(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))$$

and

$$B^k(M) = \text{Im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))$$

By (ii) above we have that $B^k(M) \subset Z^k(M)$ and the k th cohomology of M is defined as the quotient space

$$H^k(M) = Z^k(M)/B^k(M)$$

The space $H^k(M)$ measures the extent to which one cannot solve differential equations

$$\omega = d\tau$$

for $\omega \in Z^k(M)$.

Example 4.1. If $\dim(M) = n$ we have that $Z^k(M) = \Omega^k(M) = 0$ for $k > n$. Hence $H^k(M) = 0$ for $k > n$.

Example 4.2. Suppose that M is connected. Then $Z^0(M)$ consists of constant functions, i.e., we have $Z^0(M) = \mathbb{R}$. It follows that $H^0(M) = \mathbb{R}$.

Example 4.3. Suppose that $M = \mathbb{R}$. Then any $\omega \in \Omega^1(\mathbb{R})$ is on the form

$$\omega = f dt$$

with $f \in C^\infty(\mathbb{R})$. In this case we have that $Z^1(\mathbb{R}) = \Omega^1(\mathbb{R})$ (see Example 4.1). Now on $\mathbb{R}$ we know that any smooth f has an antiderivative, i.e., there exists $g \in C^\infty(\mathbb{R})$ such that $g' = f$. But then $dg = \omega$, and it follows that $H^1(\mathbb{R}) = 0$.

Example 4.4. We consider $M = \mathbb{R}^2 \setminus \{0\}$. Let $\omega \in \Omega^1(M)$ be defined by

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}$$

and check that $\omega \in Z^1(M)$. A calculation shows that

$$\int_{S^1} \omega = 2\pi$$

We have seen that for an admissible path $\gamma : [a, b] \rightarrow M$ and $f \in C^\infty(M)$ we have that

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$$

(The Fundamental Theorem of Analysis). Since $\gamma(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$ parametrizes S^1 and $\gamma(0) = \gamma(2\pi)$ this shows that there exists no $f \in C^\infty(M)$ such that $df = \omega$. Hence one cannot solve $df = \alpha\omega$ for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and so $[\omega]$ generates a 1-dimensional subspace L of $H^1(M)$.

It turns out that $L = H^1(M)$. If $\tilde{\omega} \in Z^1(M)$ we have that

$$\int_{S^1} \alpha\omega - \tilde{\omega} = 0$$

for some $\alpha \in \mathbb{R}$. In that case we may solve the equation $df = \alpha\omega - \tilde{\omega}$ (indicated earlier in class), and it follows that $[\tilde{\omega}] = \alpha[\omega]$. (We have seen in class, using homotopy invariance, that $H^2(M) = 0$ - M has the homotopy type of S^1 - see also below.)

4.2. Smooth maps induce homeomorphisms on cohomology. Suppose that $F : M \rightarrow N$ is a smooth map between manifolds. We have seen that there are linear maps

$$F^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

and that F^* commutes with d , i.e., $F^*d\omega = dF^*\omega$. It follows that F^* induces linear maps

$$F^\sharp : H^k(N) \rightarrow H^k(M)$$

defined by $F^\sharp([\omega]) := [F^*\omega]$. It is common to often drop the notation $F^\sharp$ and let F^* also denote the map on cohomology.

Recall that if $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth maps between manifolds we have that

$$(G \circ F)^* = F^* \circ G^*$$

It follows that also

$$(G \circ F)^\sharp = F^\sharp \circ G^\sharp$$

4.2.1. *Diffeomorphisms.* In the special case where $F : M \rightarrow N$ is a diffeomorphism, we have that F has an inverse $G : N \rightarrow M$ and we get that

$$\text{Id}_M^\# = (G \circ F)^\# = F^\# \circ G^\# \text{ and } \text{Id}_N^\# = (F \circ G)^\# = G^\# \circ F^\#$$

and so we see that $F^\#$ is an isomorphism.

4.2.2. *Homotopy Invariance.* Let $f, h : M \rightarrow N$ be smooth maps between manifolds. It turns out that if f and h are smoothly homotopic, i.e., there exists a smooth $F : M \times \mathbb{R} \rightarrow N$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = h$, then they induce the same map on cohomology

$$f^\# = h^\# : H^*(N) \rightarrow H^*(M)$$

This can be used to generalize the isomorphism statement above to a much more general setting than a diffeomorphism. Recall that $f : M \rightarrow N$ is a *homotopy equivalence* if there exists a smooth map $g : N \rightarrow M$ (a *homotopy inverse*) such that both $g \circ f$ and $f \circ g$ are smoothly homotopic to the identity map. It would follow that both $(g \circ f)^\#$ and $(f \circ g)^\#$ are the identity maps on $H^*(M)$ and $H^*(N)$ respectively. We have that

$$(g \circ f)^\# = f^\# \circ g^\# \text{ and } (f \circ g)^\# = g^\# \circ f^\#$$

and it follows that $f^\#$ is an isomorphism.

Example 4.5. Consider the inclusion map $\iota : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\} = M$. We claim that the map $g(x) = x/\|x\|$ is a homotopy inverse to ι . We have that $g \circ \iota = \text{Id}$.

Define $F : M \times \mathbb{R} \rightarrow M$ by

$$F(x, t) = e^{(1-t) \log \|x\|} x / \|x\|$$

Then $F(\cdot, 0) = \text{Id}$ and $F(\cdot, 1) = \iota \circ g$. We conclude that $H^2(M) = 0$, since $H^2(S^1) = 0$ by reason of dimension.

4.3. The Long Exact Sequence on Cohomology. Suppose we have a short exact sequence of cochain complexes

$$0 \rightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \rightarrow 0$$

This means that we have the following commutative diagram.

$$\begin{array}{ccccccc}
& & d \uparrow & & d \uparrow & & d \uparrow \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{i} & B_{k+1} & \xrightarrow{j} & C_{k+1} \longrightarrow 0 \\
& & d \uparrow & & d \uparrow & & d \uparrow \\
0 & \longrightarrow & A_k & \xrightarrow{i} & B_k & \xrightarrow{j} & C_k \longrightarrow 0 \\
& & d \uparrow & & d \uparrow & & d \uparrow \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{i} & B_{k-1} & \xrightarrow{j} & C_{k-1} \longrightarrow 0 \\
& & d \uparrow & & d \uparrow & & d \uparrow
\end{array}$$

We have seen that there is an induced long exact sequence on cohomology

$$\cdots \rightarrow H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \rightarrow \cdots$$

This is useful for computing cohomology groups; see the simple Example 4.6 below .

4.4. The Mayer-Vietoris Sequence. Let M be a smooth manifold and $M = U \cup V$ for open sets U and V . Then there are maps

$$i : \Omega^k(M) \rightarrow \Omega^k(U) \oplus \Omega^k(V)$$

and

$$j : \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V)$$

defined by

$$i(\omega) = (\omega|_U, \omega|_V) \text{ and } j(\omega, \tau) = \tau|_{U \cap V} - \omega|_{U \cap V}$$

Then

$$0 \rightarrow \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \rightarrow 0$$

is a short exact sequence of cochain complexes.

Example 4.6. We consider S^1 covered by two connected open "intervals" U, V such that $U \cap V$ is two disjoint "intervals" (see Fig. 26.2 in the course book). We get that the induced long exact sequence is

$$0 \rightarrow \mathbb{R} \xrightarrow{i^*} \mathbb{R} \times \mathbb{R} \xrightarrow{j^*} \mathbb{R} \times \mathbb{R} \xrightarrow{d^*} H^1(S^1) \rightarrow 0 \rightarrow \cdots$$

We deduce from this that $H^1(S^1) = \mathbb{R}$.

REFERENCES

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