

MAT4520 - Manifolds, Spring 2026

Solutions to assignment

Problem 1. (i) Let

$$Q : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto \langle Ax, x \rangle$$

and $q := Q|_{S^{n-1}}$. Since $q(x) = q(-x)$ there is a function f on $\mathbb{RP}^n$ such that $f \circ \pi = q$. Because π is surjective, f is unique. Because q is smooth and π is a local diffeomorphism, f is also smooth.

(ii) Because π is a local diffeomorphism, the critical points of f are the images of critical points of q . To determine the latter, notice that the derivative of Q at a point x is given by

$$DQ(x)y = 2\langle Ax, y \rangle.$$

It follows that a point $x \in S^{n-1}$ is a critical point of q if and only if Ax is perpendicular to the tangent space $T_x S^{n-1}$, or equivalently, if x is an eigenvalue of A .

By assumption, A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$, so for each i the eigenspace of λ_i has the form $\mathbb{R}e_i$ for some $e_i \in S^{n-1}$. Since $\mathbb{R}e_i \cap S^{n-1} = \{e_i, -e_i\}$, the function f has exactly n critical points, namely $\pi(e_1), \dots, \pi(e_n)$.

Problem 2. Let U be the open subset of M consisting of all matrices given in block form as

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an invertible $r \times r$ matrix. We first show that $S \cap U$ is a submanifold of M . For any natural number j let I_j denote the $j \times j$ identity matrix. Let E be as above and

$$F := \begin{pmatrix} I_r & 0 \\ -CA^{-1} & I_{m-r} \end{pmatrix}.$$

Then F is invertible and

$$FE = \begin{pmatrix} A & B \\ 0 & \alpha(E) \end{pmatrix},$$

where

$$\alpha(E) = -CA^{-1}B + D.$$

Therefore, E has rank r if and only if $\alpha(E) = 0$. But α is a submersion from U onto the space of $(m-r) \times (n-r)$ matrices (as one sees by varying D), so $S \cap U = \alpha^{-1}(0)$ is a submanifold of U (hence of M) of codimension $(m-r)(n-r)$.

If E is an arbitrary point in S then suitable permutations of the rows and columns of E will yield an element of U . Since any such permutation defines a diffeomorphism of M which maps S into itself it follows that S is a submanifold of M of dimension

$$\dim M = mn - (m-r)(n-r) = r(m+n-r).$$

Problem 3.

By Equation (14.7), a curve

$$\gamma(t) = (f(t), g(t))$$

is an integral curve of V if and only if

$$f'(t) = g(t), \quad g'(t) = 1,$$

or equivalently,

$$g(t) = t + b, \quad f(t) = \frac{t^2}{2} + bt + a$$

for some constants $a, b \in \mathbb{R}$. This shows that V is complete, and its flow is

$$\Phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (t, (a, b)) \mapsto \left(\frac{t^2}{2} + bt + a, t + b \right).$$

Problem 4. (i) We show that G is an affine subspace of $M(3 \times 3, \mathbb{R})$ and therefore a regular submanifold.

Let W be any n -dimensional real vector space and $V \subset W$ a k -dimensional linear subspace. Let $p \in W$ and consider the affine space

$$E := p + V = \{p + v : v \in V\}.$$

Choose a basis $v_1, \dots, v_k$ for V and extend it to a basis $v_1, \dots, v_n$ for W . Let $T : W \rightarrow \mathbb{R}^n$ be the linear map such that $Tv_i = e_i$, where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. Define

$$\phi : W \rightarrow \mathbb{R}^n, \quad x \mapsto T(x - p).$$

Then

$$T(E) = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n.$$

This shows that E is a regular submanifold of W .

To apply this to the problem at hand, let $W := M(3 \times 3, \mathbb{R})$, and let V be the space of upper triangular 3×3 matrices with zeros on the diagonal. Let $p \in W$ be the identity matrix. Then $E = G$, hence G is a regular submanifold of $M(3 \times 3, \mathbb{R})$.

(ii) We first show that G is an abstract subgroup of the general linear group $\text{GL}(3, \mathbb{R})$. For any real numbers a, b, c, a', b', c' we have

$$\begin{aligned} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & a+a' & b'+ac'+b \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

This shows that G is closed under the operations of taking products and inverses. Since G is also non-empty, it follows that G is a subgroup of $\text{GL}(3, \mathbb{R})$. Combining this with the result of (i), it follows from Proposition 15.11 that G is a Lie group.

(iii) From the proof of (i) we see that the map

$$\phi : \mathbb{R}^3 \rightarrow G, \quad (a, b, c) \mapsto \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is a diffeomorphism. The derivative of ϕ at $0 \in \mathbb{R}^3$ maps the standard basis for $\mathbb{R}^3$ to v_1, v_2, v_3 , where

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Identifying $T_I G$ with its image in $T_I \text{GL}(3, \mathbb{R})$ and the latter with $M(3 \times 3, \mathbb{R})$, we conclude that v_1, v_2, v_3 is a basis for $T_I G$.

(iv) Applying Propositions 16.11 and 16.14 we calculate

$$[v_1, v_2] = 0, \quad [v_1, v_3] = v_2, \quad [v_2, v_3] = 0.$$