MANDATORY ASSIGNMENT FOR MAT4530 SPRING 2016

Turn in your written answers before Thursday April 28th 2016 at 14:30, in the assignment box on the 7th floor of Niels Henrik Abel's house. There are often long queues just before the deadlines, so you are adviced to turn in your paper early. Remember to use the official front page. If you need a delayed deadline, due to illness or other circumstances, you must apply for an extension to Helena Båserud Mathisen (room B718, NHA, e-mail: studieinfo@math.uio.no, phone 22 85 59 07). Remember that illness has to be documented by a medical doctor. See `http://www.mn.uio.no/math/english/studies/admin/mandatory-assignments/index.html` for more information about rules concerning compulsory assignments at the Department of Mathematics. The assignment is mandatory, and students who do not get their paper accepted will not get access to the final exam. To get this paper accepted you must answer at least 50% of (the six parts 1(a) through 2(d) of) the problem set correctly. You may get partial credit for partial solutions, so turn in all your work.

Problem 1

For any group homomorphism $f\colon A \to B$, let $\ker(f) = \{a \in A \mid f(a) = 0\}$ be its kernel, let $\operatorname{im}(f) = \{f(a) \in B \mid a \in A\}$ be its image, and let $\operatorname{cok}(f) = B/\operatorname{im}(f)$ be its cokernel. Consider a commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0\,.
\end{array}$$

Assume that each row is *exact*, meaning that f is injective, $\operatorname{im}(f) = \ker(g)$ and g is surjective, and similarly for f' and g'.

(a) Watch the initial segment with Jill Clayburgh from "It's my turn" (1980)

`http://www.imdb.com/title/tt0080936/`

on

`http://www.math.harvard.edu/~knill/mathmovies/swf/myturn_snake.html`

or

`https://www.youtube.com/watch?v=etbcKWEKnvg` .

Write down the definition of a well-defined function

$$s\colon \ker(\gamma) \longrightarrow \operatorname{cok}(\alpha)$$

following Clayburgh's explanation, taking the student's objections into account.

(b) Prove that s is a group homomorphism, i.e., that $s(c_1 + c_2) = s(c_1) + s(c_2)$ for any two elements $c_1, c_2 \in \ker(\gamma)$.

Problem 2

(a) Explain why three points in the plane,

$$A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2) \in \mathbb{R}^2\,,$$

determine a positively oriented, *labeled triangle* ΔABC if and only if

$$(b_1 - a_1)(c_2 - b_2) > (b_2 - a_2)(c_1 - b_1)\,.$$

Hint: Consider the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ considered as vectors in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Date: April 4th 2016.

(b) Let

$$Y = \{(A, B, C) \in (\mathbb{R}^2)^3 \mid (b_1 - a_1)(c_2 - b_2) > (b_2 - a_2)(c_1 - b_1)\}$$

be the space of all positively oriented, labeled triangles, viewed as an open subspace of $(\mathbb{R}^2)^3$. Let $Z \subset Y$ be the subspace of points (A, B, C) where $A = (0, 0)$, $\overrightarrow{AB}$ has length 1, $\angle ABC = \pi/2$ and $\overrightarrow{BC}$ has length 1.

Show that Z is a deformation retract of Y. Use this to calculate the fundamental group $\pi_1(Y, y_0)$, where $y_0 = ((0, 0), (1, 0), (1, 1))$ is a base point in (Z and in) Y. Hint: Use Theorem 1.7 and Proposition 1.17.

(c) Let

$$X = Y/\sim,$$

where $(A, B, C) \sim (B, C, A) \sim (C, A, B)$, so that a point in X is an *unlabeled triangle*. In other words, X is the orbit space

$$X = Y/G$$

where $G = \{e, g, g^2\}$ is the cyclic group of order 3 and $g \in G$ acts on Y by sending (A, B, C) to (B, C, A).

Show that the canonical map

$$p\colon Y \longrightarrow X$$

(that forgets the labeling) is a covering space. Hint: Use Proposition 1.40.

(d) Let $x_0 = p(y_0)$. Determine the structure of the fundamental group $\pi_1(X, x_0)$, and describe the homomorphism $p_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ in algebraic terms.

— END —