HOMOLOGY

JOHN ROGNES

1. Homology

1.1. The Euler characteristic. The *Euler characteristic* of a compact triangulated surface X is defined to be the alternating sum

$$\chi(X) = V - E + F$$

where V, E and F are the number of vertices, edges and faces (= triangles) of the triangulation. It is a *homotopy invariant*, in the sense that if Y is another compact triangulated surface, and there exists a homotopy equivalence $X \simeq Y$, then $\chi(X) = \chi(Y)$, even if the number of vertices, edges and faces of the triangulation of Y may all differ from those for X. This is an interesting fact already when X and Y are topologically equivalent, i.e., when there exists a homeomorphism $X \cong Y$. The Euler characteristic is also *additive*, in the sense that if A and B are subcomplexes of X, with union $A \cup B$ and intersection $A \cap B$, then

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)\,.$$

Equivalently,

$$\chi(B) - \chi(A \cap B) = \chi(A \cup B) - \chi(A)$$

expresses the fact that $B \setminus (A \cap B)$ is equal to $(A \cup B) \setminus A$, even if these setwise differences are usually not subcomplexes of X. Additivity allows us to determine the Euler characteristic of $A \cup B$ from knowledge of the Euler characteristics of smaller subcomplexes, which allows for an inductive calculation of this invariant.

Exercise: Calculate the Euler characteristics of the boundary surface of a tetrahedron, $\partial\Delta^3$, and of an icosahedron. Write the tetrahedron as $A \cup B$ where A is the union of two of its faces and B is the union of the remaining two faces. Calculate the Euler characteristics of A, B and $A \cap B$, and verify the relations above.

Exercise: Calculate the Euler characteristic of a closed orientable surface of genus g, and of a closed non-orientable surface of genus h.

1.2. Homology groups. A finitely generated abelian group A contains a maximal finite subgroup, namely the group $\operatorname{tor} A$ of elements of finite order, and the quotient group $A/\operatorname{tor} A$ is isomorphic to $\mathbb{Z}^r$ for a well-defined non-negative integer $r = \operatorname{rk} A$ called the *rank* of A. The *homology groups* of the compact triangulated surface X are finitely generated abelian groups $H_0(X)$, $H_1(X)$ and $H_2(X)$, such that

$$\chi(X) = \operatorname{rk} H_0(X) - \operatorname{rk} H_1(X) + \operatorname{rk} H_2(X)\,.$$

Like $\chi(X)$, but unlike the numbers V, E and F associated to a given triangulation, they are homotopy invariant: if $X \simeq Y$ then there are isomorphisms $H_0(X) \cong H_0(Y)$, $H_1(X) \cong H_1(Y)$ and $H_2(X) \cong H_2(Y)$ of abelian groups. Hence also the ranks $\operatorname{rk} H_0(X)$, $\operatorname{rk} H_1(X)$ and $\operatorname{rk} H_2(X)$ are homotopy invariant, which of course implies that the alternating sum $\chi(X)$ is homotopy invariant. These ranks are also known as the *Betti numbers* of X.

Knowledge of the homology groups $H_n(X)$ for $n = 0$, 1 and 2 is therefore more precise information about X than knowledge of the Euler characteristic. In fact, the ranks of the homology groups give lower bounds for the numbers of vertices, edges and faces in any triangulation:

$$\operatorname{rk} H_0(X) \le V \quad , \quad \operatorname{rk} H_1(X) \le E \quad \text{and} \quad \operatorname{rk} H_2(X) \le F\,. \tag{1}$$

We shall study the homology groups as a finer invariant of the homotopy type of X than the Euler characteristic, answering more refined questions about spaces in that homotopy type. The n-th homology group, $H_n(X)$, is in a sense concerned with n-dimensional aspects of X, for each $n \ge 0$.

Date: March 30th 2016.

The homology groups are also additive, but in a more subtle sense than the Euler characteristic: There is a so-called *exact sequence*

$$0 \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B)$$
$$\xrightarrow{\partial} H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(A \cup B)$$
$$\xrightarrow{\partial} H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$$

for subcomplexes A and B of X. We will explain later what this means for this diagram to be an exact sequence, but, like the additivity property of the Euler characteristic, this property often allows for an inductive calculation of the homology groups of $A \cup B$ from knowledge of the homology groups of the smaller subcomplexes A, B and $A \cap B$. The fact that homology groups can often be computed or analyzed is an important contributing factor to their usefulness.

From a slightly different point of view, the additivity can be expressed in terms of so-called *relative homology groups* $H_n(X, A)$, for pairs (X, A) with $A \subset X$, with *excision* isomorphisms

$$H_n(B, A \cap B) \cong H_n(A \cup B, A)$$

for $n = 0$, 1 and 2, and exact sequences

$$0 \to H_2(A) \to H_2(X) \to H_2(X, A) \xrightarrow{\partial} H_1(A) \to H_1(X) \to H_1(X, A)$$
$$\xrightarrow{\partial} H_0(A) \to H_0(X) \to H_0(X, A) \to 0\,.$$

We shall define homology groups $H_n(X)$ not just for triangulated surfaces and integers $0 \le n \le 2$, but for general topological spaces X and arbitrary (usually non-negative) integers n. This will be the *singular homology theory* first introduced in this generality by Samuel Eilenberg in 1944, but with a history going back to Henri Poincaré in the 1890s. Before we have established the formal properties satisfied by the (singular) homology groups, it may be a little difficult to understand the scope of the general definition. It may therefore be interesting to look at other constructions of the homology groups, for special classes of topological spaces, that depend upon (and take advantage of) particular properties of the spaces involved.

1.3. **Simplicial homology.** One special class of spaces is that of *ordered simplicial complexes.* To each ordered simplicial complex K we can associate homology groups $H_n^\Delta(K)$ for each integer n (the groups for $n < 0$ are zero), and there is a natural isomorphism

$$H_n^\Delta(K) \cong H_n(X)$$

where $X = |K|$ is the topological space given by the realization of the simplicial complex.

We illustrate this in the case of a 2-dimensional simplicial complex, such as that occurring in a triangulation of a surface. Such a complex K is a collection of vertices v, edges e and triangles t in an ambient Euclidean space $\mathbb{R}^N$. These are collectively called the simplices in K. We assume that each face of any simplex in K is again a simplex in K. Furthermore we assume that the intersection of any two simplices in K is either empty or a common face of the two simplices.

In an ordered simplicial complex, we assume that the vertices of any simplex, i.e., the two end-points of any edge, and the three corners of any triangle, have been totally ordered. Hence the two end-points of an edge e have been enumerated as v_0 and v_1, while the three corners of a triangle t have been enumerated as v_0, v_1 and v_2. We assume that the given ordering of any triangle restricts to the given ordering on each of its three edges. Hence, to each triangle t with corners ordered as v_0, v_1 and v_2, the edge e_0 opposite to v_0 has end-points ordered as v_1 and v_2, the edge e_1 opposite to v_1 has end-points ordered as v_0 and v_2, and the edge e_2 opposite to v_2 has end-points ordered as v_0 and v_1.

Now let

$$\Delta_0(K) = \mathbb{Z}\{v \mid v \text{ is a vertex of } K\}$$
$$\Delta_1(K) = \mathbb{Z}\{e \mid e \text{ is an edge of } K\}$$
$$\Delta_2(K) = \mathbb{Z}\{t \mid t \text{ is a triangle of } K\}$$

be free abelian groups of ranks V, E and F, respectively. (In this context, each face is a triangle.) Let

$$\partial_1 \colon \Delta_1(K) \longrightarrow \Delta_0(K)$$

be the homomorphism given on the generators by

$$\partial_1(e) = v_1 - v_0\,.$$

Here $v_1 - v_0$ is the difference in the abelian group $\Delta_0(K)$ between the two generators corresponding to v_0 and v_1, where these vertices are the two end-points of the edge e, in the preferred ordering. By linearity, this determines the homomorphism ∂_1 on arbitrary finite sums of edges with integer coefficients. Similarly, let

$$\partial_2 \colon \Delta_2(K) \longrightarrow \Delta_1(K)$$

be the homomorphism given on the generators by

$$\partial_2(t) = e_0 - e_1 + e_2 \,.$$

Here e_0, e_1 and e_2 are the three edges of the triangle t, opposite to the corners v_0, v_1 and v_2, respectively. Again, the condition of linearity specifies hos ∂_2 is defined on arbitrary elements of $\Delta_2(K)$. We draw these groups and homomorphisms as follows:

$$0 \to \Delta_2(K) \xrightarrow{\partial_2} \Delta_1(K) \xrightarrow{\partial_1} \Delta_0(K) \to 0 \,. \tag{2}$$

Let us calculate the composite

$$\partial_1 \circ \partial_2 \colon \Delta_2(K) \longrightarrow \Delta_0(K) \,.$$

It is determined by its value on the triangles t generating $\Delta_2(K)$, and

$$(\partial_1 \circ \partial_2)(t) = \partial_1(\partial_2(t)) = \partial_1(e_0 - e_1 + e_2) = \partial_1(e_0) - \partial_1(e_1) + \partial_1(e_2)$$

by linearity. Here $\partial_1(e_0) = v_2 - v_1$, $\partial_1(e_1) = v_2 - v_0$ and $\partial_1(e_2) = v_1 - v_0$, so

$$\partial_1(e_0) - \partial_1(e_1) + \partial_1(e_2) = (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0 \,.$$

Hence

$$\partial_1 \circ \partial_2 = 0 \,.$$

We express this by saying that the diagram (2) is a *chain complex.* (Any homomorphism from 0, and any homomorphism to 0, must necessarily be the zero homomorphism, so the two remaining composites in the diagram are also zero.)

Example 1.1. Let K be the subcomplex of the tetrahedron Δ^3 with vertices a, b, c and d, generated by the faces abc and bcd and the edge ad. It consists of all the vertices a, b, c and d, all the edges ab, ac, ad, bc, bd and cd, and two of the triangles abc and bcd of the boundary of $abcd$. In each simplex we order the vertices as a subset of $\{a, b, c, d\}$, with $a < b < c < d$. Then

$$\begin{aligned}
\Delta_0(K) &= \mathbb{Z}\{a, b, c, d\} \\
\Delta_1(K) &= \mathbb{Z}\{ab, ac, ad, bc, bd, cd\} \\
\Delta_2(K) &= \mathbb{Z}\{abc, bcd\}
\end{aligned}$$

while

$$\partial_2(abc) = bc - ab + ab \quad , \quad \partial_2(bcd) = cd - bd + bc$$

and

$$\begin{aligned}
\partial_1(ab) &= b - a \,, \quad \partial_1(ac) = c - a \,, \quad \partial_1(ad) = d - a \,, \\
\partial_1(bc) &= c - b \,, \quad \partial_1(bd) = d - b \,, \quad \partial_1(cd) = d - c \,.
\end{aligned}$$

Note that $\partial_1(\partial_2(abc)) = 0$ and $\partial_1(\partial_2(bcd)) = 0$.

Let $\mathrm{im}(\partial_1) \subset \Delta_0(K)$ be the image of the homomorphism ∂_1, as a subgroup of $\Delta_0(K)$. It consists of the finite sums with integer coefficients of expressions like $v_1 - v_0$, where v_0 and v_1 are the end-points of a common edge e.

Let $\mathrm{im}(\partial_2) \subset \Delta_1(K)$ be the image of the homomorphism ∂_2, as a subgroup of $\Delta_1(K)$. It consists of the finite sums with integer coefficients of expressions like $e_0 - e_1 + e_2$, where where e_0, e_1 and e_2 are the (ordered) edges of a common triangle t.

Let $\ker(\partial_1) \subset \Delta_1(K)$ be the kernel of the homomorphism ∂_1, also viewed as a subgroup of $\Delta_1(K)$. It consists of the finite sums with integer coefficients of edges e, such that in the corresponding sum of differences $v_1 - v_0$, each vertex v of K occurs algebraically zero times.

Let $\ker(\partial_2) \subset \Delta_2(K)$ be the kernel of the homomorphism ∂_2, as a subgroup of $\Delta_2(K)$. It consists of the finite sums with integer coefficients of triangles t, such that in the corresponding sum of differences $e_0 - e_1 + e_2$, each edge e of K occurs algebraically zero times.

Since $\partial_1 \circ \partial_2 = 0$, and element of the form $\partial_2(t)$ is mapped to zero by ∂_1. In other words, the image of ∂_2 is contained in the kernel of ∂_1, both being subgroups of $\Delta_1(K)$:

$$\mathrm{im}(\partial_2) \subset \ker(\partial_1) \,.$$

The *simplicial homology groups* of K are now defined as follows:

$$H_0^\Delta(K) = \Delta_0(K)/\operatorname{im}(\partial_1)$$
$$H_1^\Delta(K) = \ker(\partial_1)/\operatorname{im}(\partial_2)$$
$$H_2^\Delta(K) = \ker(\partial_2)\,.$$

Hence $H_0^\Delta(K)$ is the free abelian group generated by the vertices v of K, modulo the subgroup generated by the differences $v_1 - v_0$ where v_0 and v_1 are the end-points of an edge e. It follows that $H_0^\Delta(K)$ is isomorphic to the free abelian group on the set of path components of $X = |K|$, since two vertices are in the same path component if and only if they can be connected by a finite chain of edges.

More subtly, $H_1^\Delta(K)$ measures the group-theoretic difference between the (stronger) condition of being in the image of ∂_2 and the (weaker) condition of being in the kernel of ∂_1. If these conditions are equivalent for K then $H_1^\Delta(K) = 0$ is the trivial group, and vice versa. If $X = |K|$ is path-connected, with a base-point x_0, then Poincaré proved that there is a natural isomorphism $\pi_1(X, x_0)^{ab} \cong H_1^\Delta(K)$, where $\pi = \pi_1(X, x_0)$ is the fundamental group of X and $\pi^{ab} = \pi/[\pi, \pi]$ is the *abelianization* of π, i.e., the quotient of π by the (automatically normal) subgroup of π generated by all commutators $[a, b] = aba^{-1}b^{-1}$ for $a, b \in \pi$.

Finally, $H_2^\Delta(K)$ consists of 2-*cycles* on K, i.e., finite signed sums of triangles such that each edge e that occurs in $\partial_2(t)$ as an edge of a triangle t also occurs with the opposite sign in $\partial_2(\pm t')$ as an edge of another triangle t' (or its negative). If K triangulates an oriented closed (and connected) surface, then $H_2^\Delta(K) \cong \mathbb{Z}$, with one generator corresponding to the signed sum of all the triangles t in K, with the sign $+$ if the loop from v_0 via v_1 to v_2 and back to v_0 is positively oriented, and the sign $-$ otherwise. This generator of $H_2^\Delta(K)$ is called the *fundamental class* of K, with respect to the given orientation. (The opposite orientation corresponds to the negative of the first generator, i.e., to the other possible generator of this infinite cyclic group.) If the surface is non-orientable, then $H_2^\Delta(K) = 0$.

Example 1.2. For the subcomplex K of Δ^3 discussed in the previous example, $H_0^\Delta(K) = \mathbb{Z}$ generated by $a \equiv b \equiv c \equiv d$, $H_1^\Delta(K) = \mathbb{Z}$ generated by $ab - ad + bd \equiv ac - ad + cd$, and $H_2^\Delta(K) = 0$.

Exercise: Calculate the homology groups of the boundary of a tetrahedron and of an icosahedron.

Exercise: Prove the inequalities (1) for a surface triangulated by a simplicial complex.

1.4. **De Rham cohomology.** Another special class of spaces is that of smooth manifolds. In 1931, Georges de Rham constructed, for each C^∞ differentiable manifold M of dimension d, real vector spaces $H^n_{dR}(M)$ for $0 \le n \le d$ that are dual to the homology groups $H_n(M)$, in the sense that there are (natural) isomorphisms

$$H^n_{dR}(M) \cong \operatorname{Hom}(H_n(M), \mathbb{R})\,.$$

If the abelian group $H_n(M)$ has rank r, then $\operatorname{Hom}(H_n(M), \mathbb{R}) \cong \mathbb{R}^r$, so the real vector space $H^n_{dR}(M)$ has dimension r. To reflect the dualization involved in passing from $H_n(M)$ to $\operatorname{Hom}(H_n(M), \mathbb{R})$, the vector spaces $H^n_{dR}(M)$ are called de Rham *cohomology* groups (or spaces).

We illustrate de Rham's theory in the case when $M = U \subset \mathbb{R}^3$ is an open subset of 3-space. We write (x, y, z) for the coordinates of that space. Let

$$C^\infty(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}$$

be the vector space of all real smooth functions f on U. Let

$$\mathscr{X}(U) = \{X : U \to \mathbb{R}^3 \mid X \text{ is a smooth vector field}\}$$

be the vector space of all smooth vector fields $X = (P, Q, R)$ on U. Let

$$\operatorname{grad} : C^\infty(U) \longrightarrow \mathscr{X}(U)$$

be given by

$$\operatorname{grad}(f) = \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big)\,,$$

mapping a function to its gradient vector field. Let

$$\operatorname{curl} : \mathscr{X}(U) \longrightarrow \mathscr{X}(U)$$

be given by

$$\operatorname{curl}(P, Q, R) = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,,$$

mapping a vector field to its curl vector field. Lastly, let

$$\operatorname{div}\colon \mathscr{X}(U) \longrightarrow C^\infty(U)$$

be given by

$$\operatorname{div}(A, B, C) = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\,,$$

mapping a vector field to its divergence function. We draw these real vector spaces and homomorphisms as follows:

$$0 \to C^\infty(U) \overset{\text{grad}}{\longrightarrow} \mathscr{X}(U) \overset{\text{curl}}{\longrightarrow} \mathscr{X}(U) \overset{\text{div}}{\longrightarrow} C^\infty(U) \to 0\,. \tag{3}$$

By the symmetry of second derivatives, also known as the equality of mixed partials, it follows that

$$\operatorname{curl} \circ \operatorname{grad} = 0 \quad \text{and} \quad \operatorname{div} \circ \operatorname{curl} = 0\,. \tag{4}$$

Hence diagram (3) is also a chain complex, although when we reinterpret it in a way that emphasizes the integer index n, it will be more natural to call it a cochain complex, or a cocomplex.

The *de Rham cohomology groups* of U are now defined as follows:

$$\begin{aligned} H^0_{dR}(U) &= \ker(\operatorname{grad}) \\ H^1_{dR}(U) &= \ker(\operatorname{curl})/\operatorname{im}(\operatorname{grad}) \\ H^2_{dR}(U) &= \ker(\operatorname{div})/\operatorname{im}(\operatorname{curl}) \\ H^3_{dR}(U) &= C^\infty(U)/\operatorname{im}(\operatorname{div})\,. \end{aligned}$$

The vector space $H^0_{dR}(U)$ consists of smooth functions f with $\operatorname{grad}(f) = 0$, i.e., the locally constant functions. If U is path connected, these are the same as the constant functions, so that $H^0_{dR}(U) \cong \mathbb{R}$. If U is not path connected, as in the example

$$U = \{(x, y, z) \mid x \neq 0\}$$

given by the complement of the yz-plane, there are more locally constant functions than the constant ones. In general $H^0_{dR}(U)$ is the product of one copy of $\mathbb{R}$ for each path component of U.

The Poincaré lemma tells us that if U is a convex subset of $\mathbb{R}^3$, then $H^n_{dR}(U) = 0$ for each $n \geq 1$, i.e., that $\operatorname{im}(\operatorname{grad}) = \ker(\operatorname{curl})$, $\operatorname{im}(\operatorname{curl}) = \ker(\operatorname{div})$ and $\operatorname{im}(\operatorname{div}) = C^\infty(U)$. Conversely, if $H^n_{dR}(U) \neq 0$, the corresponding inclusion is proper, i.e., not the identity.

Example 1.3. The vector space $H^1_{dR}(U)$ consists of curl-free vector fields, i.e., vector fields $X = (P, Q, R)$ with $\operatorname{curl}(X) = 0$, modulo gradients, i.e., vector fields of the form $X = \operatorname{grad}(f)$ for a smooth function f on U. If

$$U = \{(x, y, z) \mid (x, y) \neq (0, 0)\}$$

is the complement of the z-axis, then the vector field

$$X = \Big(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\Big)$$

satisfies $\operatorname{curl}(X) = 0$ (proof by calculation), but there is no smooth function f with $X = \operatorname{grad}(f)$, because then the line integral

$$\int_C X \cdot dr = \int_0^{2\pi} (-\sin(t), \cos(t), 0) \cdot (-\sin(t), \cos(t), 0)\, dt = 2\pi$$

along the curve $C \subset U$, with parametrization $r(t) = (\cos(t), \sin(t), 0)$ for $0 \leq t \leq 2\pi$, would have been 0 by Stokes' theorem

$$\int_C \operatorname{grad}(f) \cdot dr = \int_{\partial C} f = 0\,.$$

So X represents a nonzero class in $H^1_{dR}(U)$, which in fact gives a basis for this one-dimensional vector space. In general, a curl-free vector field X represents a class in $H^1_{dR}(U)$, which is zero if and only if X is a gradient field.

Example 1.4. The vector space $H^2_{dR}(U)$ consists of divergence-free vector fields, i.e., vector fields $Y = (A, B, C)$ with $\operatorname{div}(Y) = 0$, modulo curls, i.e., vector fields of the form $Y = \operatorname{curl}(X)$ for a smooth vector field X on U. If

$$U = \{(x, y, z) \mid (x, y, z) \neq (0, 0, 0)\}$$

is the complement of the origin, then the vector field

$$Y = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

satisfies $\operatorname{div}(Y) = 0$ (proof by calculation), but there is no smooth vector field X with $Y = \operatorname{curl}(X)$, because then the surface integral

$$\int_{S^2} Y \cdot d\Sigma = \cdots = \int_0^{2\pi} \int_0^{\pi} \sin(s)\, ds\, dt = 4\pi$$

along the unit sphere $S^2 \subset U$, with parametrization $x(s,t) = (\sin(s)\cos(t), \sin(s)\sin(t), \cos(s))$ for $0 \le s \le \pi$ and $0 \le t \le 2\pi$, would have been 0 by Stokes' theorem

$$\int_{S^2} \operatorname{curl}(X) \cdot d\Sigma = \int_{\partial S^2} Y \cdot dr = 0\,.$$

So Y represents a nonzero class in $H^2_{dR}(U)$, which in fact gives a basis for this one-dimensional vector space. In general, a divergence-free vector field Y represents a class in $H^2_{dR}(U)$, which is zero if and only if Y is a curl.

For $U \subset \mathbb{R}^3$, $H^3_{dR}(U)$ is always zero. Every smooth function can be written as the divergence of a vector field.

These arguments should illustrate how integration of differential forms is a useful analytical tool for the study of de Rham cohomology. This model for (co-)homology is therefore well adapted to the study of differential forms that arise from geometry, such as the curvature of a connection on a vector bundle. This leads to the so-called Chern–Weil theory of characteristic classes, see e.g. Appendix C in the book by Milnor and Stasheff.

To generalize these definitions to general smooth manifolds, it is convenient to rewrite the diagram (3) in terms of differential forms and exterior derivatives. Let

$$\Omega^0(U) = \{f \colon U \to \mathbb{R}\}$$

be the vector space of smooth functions, let

$$\Omega^1(U) = \{P\, dx + Q\, dy + R\, dz \mid P, Q, R \colon U \to \mathbb{R}\}$$

be the vector space of smooth 1-forms, let

$$\Omega^2(U) = \{A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy \mid A, B, C \colon U \to \mathbb{R}\}$$

be the vector space of smooth 2-forms, and let

$$\Omega^3(U) = \{g\, dx \wedge dy \wedge dz \mid g \colon U \to \mathbb{R}\}$$

be the vector space of smooth 3-forms. Under the isomorphisms

$$\begin{aligned} C^\infty(U) &= \Omega^0(U) & f &\leftrightarrow f \\ \mathscr{X}(U) &\cong \Omega^1(U) & (P,Q,R) &\leftrightarrow P\, dx + Q\, dy + R\, dz \\ \mathscr{X}(U) &\cong \Omega^2(U) & (A,B,C) &\leftrightarrow A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy \\ C^\infty(U) &\cong \Omega^3(U) & g &\leftrightarrow g\, dx \wedge dy \wedge dz \end{aligned}$$

the cochain complex appears as

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \to 0$$

where d denotes the exterior derivative. The relations (4) then appear as the formula

$$d \circ d = 0$$

at each place in the diagram. Then n-th de Rham cohomology group $H^n_{dR}(U)$ is then the subquotient $\ker(d)/\operatorname{im}(d)$ of $\Omega^n(U)$, where

$$\operatorname{im}(d) \subset \ker(d) \subset \Omega^n(U)\,.$$

1.5. **CW complexes and cellular homology.** Let X be a CW complex, with skeletal filtration

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X\,.$$

For each $n \geq 0$ we have a pushout square

$$\begin{array}{ccc} \coprod_\alpha \partial D^n & \longrightarrow & \coprod_\alpha D^n \\ {\scriptstyle \coprod_\alpha \phi_\alpha}\downarrow & & \downarrow{\scriptstyle \coprod_\alpha \Phi_\alpha} \\ X^{(n-1)} & \longrightarrow & X^{(n)} \end{array}$$

where α ranges over the set of n-cells in X, the maps $\phi_\alpha \colon S^{n-1} = \partial D^n \to X^{(n-1)}$ are called the *attaching maps* of these n-cells, and the maps $\Phi_\alpha \colon D^n \to X^{(n)}$ are called the *characteristic maps* of these n-cells. For each $n \geq 0$, let

$$C_n^{CW}(X) = \mathbb{Z}\{\alpha \mid \alpha \text{ is an } n\text{-cell of } X\}$$

be the free abelian group generated by the set of n-cells of X. Later on, we shall be able to define homomorphisms

$$\partial_n \colon C_n^{CW}(X) \longrightarrow C_{n-1}^{CW}(X)$$

for all $n \geq 1$, satisfying

$$\partial_n \circ \partial_{n+1} = 0$$

for all $n \geq 1$. The case $n = 0$ can be included in the general case by defining $\partial_0 \colon C_0^{CW}(X) \to 0$ to be the zero homomorphism. Hence we have a chain complex

$$\cdots \to C_{n+1}^{CW}(X) \xrightarrow{\partial_{n+1}} C_n^{CW}(X) \xrightarrow{\partial_n} C_{n-1}^{CW}(X) \to \cdots \to C_1^{CW}(X) \xrightarrow{\partial_1} C_0^{CW}(X) \to 0\,.$$

In particular

$$\operatorname{im}(\partial_{n+1}) \subset \ker(\partial_n)$$

as subgroups of $C_n^{CW}(X)$, for each $n \geq 0$. Then we can define the *cellular homology* groups of X as

$$H_n^{CW}(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

for all $n \geq 0$.

As in the previous special cases, cellular homology is also isomorphic to singular homology, when it is defined. There is a natural isomorphism

$$H_n^{CW}(X) \cong H_n(X)$$

for each CW complex X.

Exercise: Give the boundary of a tetrahedron, $X = \partial\Delta^3$, a cell structure with a single 0-cell and a single 2-cell. Show that

$$H_n^{CW}(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

1.6. **Δ-complexes and simplicial homology.** For some CW-complexes X, the boundary homomorphisms $\partial_n \colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ can be defined by alternating sums, as we described in the case of ordered simplicial complexes. A convenient such class of complexes was introduced by Eilenberg and Zilber (1950), under the name of *semisimplicial complexes*. The precise terminology has changed over time, so to avoid confusion, Hatcher uses the name Δ-*complexes* for this notion, intermediate in generality between ordered simplicial complexes and CW-complexes.

In abstract terms, a Δ-complex consists of collections of simplices, and functions specifying how each face of any simplex is again a simplex, of dimension one lower.

Definition 1.5. An abstract Δ-*complex* X consists of a set X_n of n-dimensional simplices, for each $n \geq 0$, and functions

$$d_i^n \colon X_n \longrightarrow X_{n-1}$$

for each $n \geq 1$ and $0 \leq i \leq n$, called *face operators*, such that

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$$

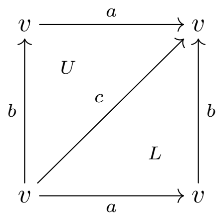
if $i < j$.

We think of each element $\sigma \in X_n$ as an n-dimensional simplex, with vertices $v_0, \dots, v_n$, enumerated in that order, starting from zero. The function d_i^n takes σ to the $(n-1)$-dimensional simplex $d_i^n(\sigma) \in X_{n-1}$ spanned by the vertices $v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ given by omitting the i-th vertex, v_i. The relation $d_i^{n-1}(d_j^n(\sigma)) = d_{j-1}^{n-1}(d_i^n(\sigma))$ for $i < j$ expresses the fact that omitting the j-th vertex and then omitting the i-th vertex leads to the same $(n-2)$-simplex as first omitting the i-th vertex and then omitting the $(j-1)$-th vertex, namely

$$v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n ,$$

since the vertex v_j became the $(j-1)$-th vertex when the vertex v_i was omitted (for $i < j$).

Example 1.6. Consider the torus obtained from a square divided along an edge c into two triangles (U and L) by identifying pairs of opposite sides (the a's and the b's).



It can be viewed as a Δ-complex X with $X_2 = \{U, L\}$, $X_1 = \{a, b, c\}$ and $X_0 = \{v\}$. The face operators $d_i^2 \colon X_2 \to X_1$ are given by

$$d_0^2(U) = a \quad , \quad d_1^2(U) = c \quad , \quad d_2^2(U) = b$$

and

$$d_0^2(L) = b \quad , \quad d_1^2(L) = c \quad , \quad d_2^2(L) = a$$

while the face operators $d_i^1 \colon X_1 \to X_0$ are given by

$$d_0^1(a) = d_1^1(a) = d_0^1(b) = d_1^1(b) = d_0^1(c) = d_1^1(c) = v .$$

Hatcher gives a more concrete formulation of the cell complex associated to an abstract Δ-complex, in §2.1. The definition of simplicial homology readily generalizes from ordered simplicial complexes to Δ-complexes:

Definition 1.7. Let X be a Δ-complex. For each integer $n \geq 0$ let

$$\Delta_n(X) = \mathbb{Z}\{\sigma \in X_n\}$$

be the free abelian group generated by the set of n-simplices of X, i.e., the abelian group of finite sums

$$\sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$$

where each coefficient n_α is an integer and each σ_α is an element of X_n. For each integer $n \geq 1$ let the homomorphism

$$\partial_n \colon \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$$

be given by the alternating sum

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i d_i^n(\sigma) .$$

This makes sense since each $d_i^n(\sigma)$ is an element of X_{n-1}.

Lemma 1.8. $\partial_{n-1} \circ \partial_n = 0$ *for each* $n \geq 2$.

Proof. Let $\sigma \in X_n$ be any generator of $\Delta_n(X)$, with $n \geq 2$.

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}(\sum_{j=0}^{n}(-1)^j d_j^n(\sigma)) = \sum_{j=0}^{n}(-1)^j \partial_{n-1}(d_j^n(\sigma)) \\
&= \sum_{j=0}^{n}(-1)^j \sum_{i=0}^{n-1}(-1)^i d_i^{n-1}(d_j^n(\sigma)) = \sum_{j=0}^{n}\sum_{i=0}^{n-1}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) \\
&= \sum_{0\leq i<j\leq n}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) + \sum_{0\leq j\leq i\leq n-1}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) \\
&= \sum_{0\leq i<j\leq n}(-1)^{i+j} d_{j-1}^{n-1}(d_i^n(\sigma)) + \sum_{0\leq j\leq i\leq n-1}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) \\
&= \sum_{0\leq j<i\leq n}(-1)^{j+i} d_{i-1}^{n-1}(d_j^n(\sigma)) + \sum_{0\leq j\leq i\leq n-1}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) \\
&= \sum_{0\leq j\leq i\leq n-1}(-1)^{j+i+1} d_i^{n-1}(d_j^n(\sigma)) + \sum_{0\leq j\leq i\leq n-1}(-1)^{i+j} d_i^{n-1}(d_j^n(\sigma)) \\
&= 0
\end{aligned}$$

by the definition of ∂_n, by linearity of ∂_{n-1}, by the definition of ∂_{n-1}, by regrouping, by splitting the double sum into the parts $i < j$ and $j \leq i$, by the assumed relation for $i < j$, by switching i and j in the first sum, by replacing i with $i + 1$ in the first sum, and the evident cancellation, respectively. □

Hence

$$\operatorname{im}(\partial_{n+1}) \subset \ker(\partial_n) \subset \Delta_n(X)$$

for each $n \geq 0$, where we interpret ∂_0 as the zero homomorphism, and the following definition makes sense.

Definition 1.9. The *simplicial chain complex* associated to the Δ-complex X is the diagram

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \cdots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \to 0$$

of abelian groups and homomorphisms. The *simplicial homology groups* of X are the homology groups of this chain complex:

$$H_n^{\Delta}(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

for $n \geq 0$, where $\partial_0 = 0$.

Example 1.10. The simplicial chain complex of the torus with the Δ-complex structure given above is

$$0 \to \mathbb{Z}\{U, L\} \xrightarrow{\partial_2} \mathbb{Z}\{a, b, c\} \xrightarrow{\partial_1} \mathbb{Z}\{v\} \to 0\,.$$

Here $\partial_2(U) = a - c + b$ is equal to $\partial_2(L) = b - c + a$, while $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$. Hence

$$\begin{aligned}
H_0^{\Delta}(X) &= \Delta_0(X)/\operatorname{im}(\partial_1) \cong \mathbb{Z}\{v\} \\
H_1^{\Delta}(X) &= \ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}\{a, b\} \\
H_2^{\Delta}(X) &= \ker(\partial_2)/0 \cong \mathbb{Z}\{L - U\}\,,
\end{aligned}$$

while $H_n^{\Delta}(X) = 0$ for $n \geq 3$. Hence the Betti numbers $\operatorname{rk} H_n^{\Delta}(X)$ are 1, 2, 1 and 0 for $n = 0$, 1, 2 and ≥ 3, respectively. In $H_1^{\Delta}(X)$, $c \equiv a + b$.

1.7. **Singular homology.** A general topological space X is not presented as a union of simplices. However, following Eilenberg one can consider the collection

$$X_n = \{\sigma \colon \Delta^n \to X\}$$

of all continuous maps $\sigma \colon \Delta^n \to X$ from a standard affine n-simplex (the convex span of the unit vectors in $\mathbb{R}^{n+1}$) to the space X, and form the free abelian group

$$S_n(X) = \mathbb{Z}\{\sigma \colon \Delta^n \to X\}$$

for each $n \geq 0$. We refer to such maps $\sigma \colon \Delta^n \to X$ as *singular simplices* in X, reflecting the fact that σ may not be an embedding, even from the interior of Δ^n. There are face operators $d_i^n \colon X_n \to X_{n-1}$ for each $0 \leq i \leq n$ and $n \geq 1$, and boundary homomorphisms

$$\partial_n \colon S_n(X) \longrightarrow S_{n-1}(X)$$

defined as in the simplicial case. The same calculation shows that $\partial_{n-1} \circ \partial_n = 0$ for $n \geq 2$, so that the diagram

$$\cdots \to S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \cdots \to S_1(X) \xrightarrow{\partial_1} S_0(X) \to 0$$

of abelian groups and homomorphisms is a chain complex, called the *singular chain complex* of X. The associated homology groups

$$H_n(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

are the *singular homology groups* of the space X.

The intermediate step in this construction, the singular complex $(S_*(X), \partial)$, usually consists of infinitely generated abelian groups, and is more difficult to enumerate than the simplicial complex associated to a finite Δ-complex. Nonetheless, it turns out that the singular homology groups are isomorphic to the simplicial homology groups when both are defined, so that the singular homology groups $H_*(X)$ have all the finiteness properties that the simplicial homology groups $H_*^{\Delta}(X)$ enjoy, even if this is less evident from the definition. The main advantage of the singular homology groups is that they are defined for arbitrary topological spaces, and are evidently topological invariants, independent of any combinatorial or smooth structure of the kind required to define simplicial homology and de Rham cohomology, respectively.

After we have spent a little more time familiarizing ourselves with Δ-complexes and simplicial homology we will therefore turn to general topological spaces and singular homology, and establish a series of formal properties of the latter theory, which allow us to show that the singular theory is a true generalization of the simplicial one.

Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no
URL: http://folk.uio.no/rognes