MAT4530 SPRING 2016 – EXAMINATION TOPICS

JOHN ROGNES

The curriculum for MAT4530 (Algebraic Topology I) is drawn from Hatcher's textbook [Hat02].

Chapter 0. Some Underlying Geometric Notions

Homotopy and Homotopy Type. Homotopy of maps, homotopic maps, relevance for later definition of fundamental group and homotopy invariance of fundamental group and homology groups, homotopy equivalences of spaces, homotopy type, retractions, deformation retractions, null-homotopic maps, convex and other contractible spaces.

Cell Complexes. (Cell=) CW complexes, skeletal filtration, attaching maps, characteristic maps, low-dimensional examples such as graphs and surfaces, higher-dimensional spheres, real and complex projective spaces.

Operations on Spaces. Products, (wedge) sums, subspaces, quotient spaces, cones, suspensions. Same for CW complexes.

Two Criteria for Homotopy Equivalence. Examples. Proofs in Proposition 0.17 and 0.18.

The Homotopy Extension Property (HEP). Example: CW pairs. Homotopy relative to a subspace. HEP equivalent to mapping cylinder a deformation retract of full cylinder.

Chapter 1. The Fundamental Group

1.1. Basic Constructions.

Paths and Homotopy. Homotopy of paths (relative to endpoints), loops, composition of paths, definition of fundamental group, verification that it is a group, weak dependence on basepoint, simply-connected spaces, contractible spaces.

The Fundamental Group of the Circle. The map $p\colon \mathbb{R} \to S^1$, relevance for later definition of covering map, unique lifting property for paths and path homotopies, relevance for later unique homotopy lifting property (HLP), calculation of $\pi_1(S^1, 1)$. Applications to fundamental theorem of algebra, Brouwer's fixed point theorem and Borsuk–Ulam theorem.

Induced Homomorphisms. Relevance for functoriality, $\pi_1(S^n, 1)$ for $n \geq 2$, relevance for van Kampen's theorem, relation to homotopy equivalences and (deformation) retractions.

1.2. Van Kampen's Theorem.

Free Products of Groups. Presentation of a group in terms of generators and relations.

The van Kampen Theorem. Precise statement of the theorem, including the special case of a union of two path-connected open sets with path-connected intersection. Rough or precise discussion of the proof. Examples, such as wedge sums of spaces, graphs, torus knot complements and a shrinking wedge of circles.

Applications to Cell Complexes. Calculation of the fundamental group of a CW complex, in terms of its 1-skeleton graph and the attaching maps of the 2-cells. Examples, such as closed surfaces, orientable and non-orientable.

1.3. Covering Spaces. Definition of a covering space $p\colon \tilde{X} \to X$. Terminology: evenly covered subsets, sheets of the covering, fiber over a point. Examples, such as covering spaces of S^1, small graphs, or $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

Date: June 1st 2016.

Lifting Properties. The (unique) homotopy lifting property (HLP) for covering spaces, including the (unique) path lifting property as a special case. The homomorphism $p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective. Description of the image in terms of lifts of loops. Role of path-connectedness for X and $\tilde{X}$. Relation between the number of sheets/elements of a fiber and the index of the image subgroup/number of cosets. Lifting criterion for maps from a path-connected and locally path-connected space. Role of local path-connectedness in the proof. Uniqueness result.

The Classification of Covering Spaces. (Free or basepoint-preserving) maps between covering spaces over a fixed base space. (Free or basepoint-preserving) isomorphism of covering spaces over a fixed base space. Galois correspondence between isomorphism classes of covering spaces over X and subgroups of the fundamental group of X. Existence of simply-connected covering space for path-connected, locally path-connected, semilocally simply-connected spaces. Precise statement, rough or detailed proof, role of semilocal simple-connectedness in proof.

Classification of connected covering spaces over a nice (path-connected, semilocally simply-connected) base space. Examples: Find covering spaces of a specific type by way of the corresponding subgroups of the fundamental group, or determine the fundamental group by knowledge of the universal covering space. Relevance for equivalence of categories between connected covering spaces with basepoint and subgroups of the fundamental group.

Omit: Subsection on "Representing Covering Spaces by Permutations".

Deck Transformations and Group Actions. The group of deck transformations. Normal (= regular) covering spaces, where the group of deck transformations acts transitively on each fiber. Relation to normal subgroups of the fundamental group. Identification of group of deck transformations with quotient group of the fundamental group. Covering space action of a group on a space, with associated covering space.

Omit: Pages 75–78, after Example 1.43.

Chapter 2. Homology

2.1. Simplicial and Singular Homology.

Δ-Complexes. Definition of affine/standard n-simplex, vertices, faces, Δ-complex (as a special CW complex, or as an abstract diagram of sets of simplices and face operator functions). Examples of low-dimensional examples such as graphs and surfaces, higher-dimensional spheres and real projective spaces.

Simplicial Homology. Simplicial n-chains, boundary homomorphism, $\partial^2 = 0$, simplicial chain complex, n-cycles, n-boundaries, homology classes, simplicial homology groups. Calculation of homology of the known examples.

Singular Homology. Definition of singular n-simplex, singular chains, faces, boundary operator, $\partial^2 = 0$, singular chain complex, n-cycles, n-boundaries, homology classes and singular homology groups. Calculation for disjoint unions, a point, and degree 0. Augmented chain complexes and reduced homology.

Homotopy Invariance. Chain complex, chain maps, induced homomorphism in homology. chain homotopies, chain homotopy classes of chain maps, equality of induced homomorphisms. Theorem: Homotopy invariance of singular homology. Precise statement. Proof in terms of prism operator, giving a chain homotopy P between chain maps $f_\#$ and $g_\#$ between singular chain complexes. Proof that $\partial P + P\partial = g_\# - f_\#$. Homotopy equivalences induce chain homotopy equivalences which are quasi-isomorphisms, i.e., induce isomorphisms in homology.

Exact Sequences and Excision. Exactness of a sequence $A \to B \to C$ of (abelian) groups. Relative homology groups of a pair (X, A). Relative n-cycles and relative n-boundaries. Short exact sequence of chain complexes. Theorem: The associated connecting homomorphism and long exact sequence of homology groups. Proof in terms of snake lemma/diagram chase. Long exact sequence in homology of a pair $A \subset X$. Long exact sequence in homology of a triple $A \subset B \subset X$

The excision theorem, precise formulation. Proposition: Subcomplex of fine singular chains (small relative to a given cover, say consisting of two subsets) is quasi-isomorphic to the full singular chain complex. Sketch or detailed proof, in terms of barycentric subdivision operators and chain homotopies to the identity, Lebesgue number of open cover. Reduction of excision theorem to this proposition.

Relation of relative homology to reduced homology of quotient for good pairs, such as CW pairs. Long exact sequence in reduced homology for a good pair.

Application to calculation of reduced homology of spheres, and relative homology of disc/boundary pairs. Application to Brouwer's fixed point theorem, invariance of dimension.

Naturality of connecting homomorphism and long exact sequence, relevance to natural transformations between functors.

The Equivalence of Simplicial and Singular Homology. Five-lemma. Simplicial/singular comparison theorem, proof by induction over skeleta. Homology of sequential unions as direct/inductive (co-)limit of finite cases.

2.2. Computations and Applications.

Degree. Formal properties of the degree of a map $f\colon S^n \to S^n$, including reflections and the antipodal map. Examples: (Non-)existence of nonzero vector fields, (Non-)existence of free actions by groups on spheres. Theorem: The local degree formula for a map $f\colon S^n \to S^n$. Precise statement, sketch or detailed proof.

Cellular Homology. Cellular chains, cellular chain complex, cellular homology. Cellular/singular comparison theorem, proof by induction over skeleta and passage to colimits. Examples where cellular boundary is zero, such as complex projective spaces. Cellular boundary formula in terms of degrees. Examples: Surfaces, spheres, real projective spaces. Euler characteristic, in terms of cells or homology groups. Homotopy invariance and additivity. Split short exact sequences.

Omit: Subsection on "Homology of groups".

Mayer–Vietoris Sequences. Theorem: Mayer–Vietoris sequence in homology for an excisive triad (X, A, B), with X the union of the interiors of A and B. Reduction to the long exact sequence in homology associated to a short exact sequence of chain complexes. Description of connecting homomorphism. Alternative proof in terms of Barratt–Whitehead lemma.

Omit: Example 2.48 and remainder of subsection.

Homology with Coefficients. Definition of homology groups with coefficients in an abelian group G. Calculation for real projective spaces.

2.3. The Formal Viewpoint.

Axioms for Homology. Definition of a homology theory on CW complexes, in terms of homology groups, induced homomorphisms and connecting homomorphism, satisfying functoriality and naturality, homotopy invariance, exactness and the wedge axiom. Role of the coefficient groups of the homology theory.

Categories and Functors. Categories, objects, morphisms, isomorphisms, commuting diagrams, functors, preservation of isomorphisms and commuting diagrams, natural transformations, natural equivalences. Examples involving (pairs of) spaces/CW complexes, Δ-complexes, chain complexes, (graded) homology groups, etc.

2.A. Homology and Fundamental Group. Proof that $\pi_1(X, x_0)_{ab} \cong H_1(X)$ for path-connected X. Examples for surfaces.

2.B. Classical Applications. Calculation of homology of complement of discs or spheres in S^n. Jordan curve theorem and generalizations. Knot theory. Invariance of domain. No continuous injection $\mathbb{R}^m \to \mathbb{R}^n$ for $m > n$.

Omit: Subsections on "Division Algebras" and "Borsuk–Ulam Theorem".

Appendix

Topology of Cell Complexes. Prop. A.5: A subcomplex of a CW complex is a deformation retract of an open neighborhood.

References

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354