Introduction.

Paradigm of algebraic topology :
allow continuous deformation of spaces, maps, ..
and see which properties remain the same

do not care about : distance, homeomorphism type, ..

care about : "number of holes / loops up to deform.", ..

tools : abstract algebra ; groups, rings, modules, ..

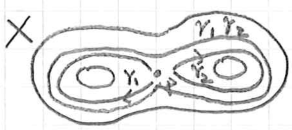
capture structures in "loops up to cont. def.", "composition of such loops" etc.

Main goals of this course :

1. <u>fundamental</u> <u>group</u> $\pi_1(X, p)$ (X ← space, p ← basepoint)

"loops in X based at p, up to cont def"

group law with path composition



2. (singular) <u>homology</u> <u>groups</u> $H_n(X)$ $n = 0, 1, 2, \cdots$

"formal linear comb. of maps $S^n \xrightarrow{\sigma} X$; $a\sigma_1 + b\sigma_2, \cdots$
up to cont. def.

3. give computable models of $\pi_1(X)$, $H_n(X)$ for spaces given by glueing basic parts

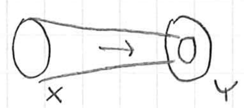
Chapter 0 : Homotopy

Want : represent continuous deformation of maps (and spaces).

Example $X = Y = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

$f_t : X \to Y$, $f_t(x,y) = (tx, ty)$ $0 \leq t \leq 1$.

$f_1 = id$, $f_0(x,y) = (0,0)$ f_t



Convention X, Y : topological spaces, $f : X \to Y$: cont. map

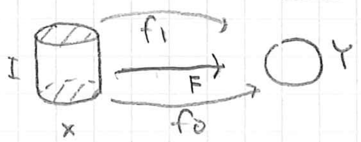
$I = [0,1]$ unit interval

Def. $f_0, f_1 : X \to Y$; f_0 and f_1 are <u>homotopic</u>

if $\exists\, F : X \times I \to Y$ s.t. $f_0(x) = F(x,0)$, $f_1(x) = F(x,1)$

write $f_0 \simeq f_1$ or $f_0 \sim_h f_1$

say F is a <u>homotopy</u> between f_0 and f_1.



Morally : F represents a cont. family $f_t : X \to Y$.

to be precise : cont map $I \to C(X, Y)$
space of cont maps with compact-open topology

"Shrink down a space"

Def $A \subset X$ is a <u>deformation</u> <u>retract</u> if

$\exists\, r : X \to A$ s.t. $\begin{cases} r(a) = a & \text{for } a \in A \\ id_X \text{ and } r \text{ are homotopic as} & \\ \text{cont. maps } X \to X \end{cases}$

Example $X = \{(x,y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2+y^2} \leq 3\}$

$A = \{(x,y) : \sqrt{x^2+y^2} = 2\}$

$r(x,y)$: "projection" from X to A

$F(x,y,t) = (1-t)\, r(x,y) + t\,(x,y)$

" Spaces up to cont. deform "

Def. X and Y are <u>homotopy equivalent</u> if
$\exists\ f: X \to Y,\ g: Y \to X$ s.t.
1. fg and id_Y are homotopic (as maps $Y \to Y$)
2. gf and id_X are homotopic. (as $X \to X$)

write $X \simeq Y$, $X \simeq_h Y$, etc.

Example $A \subset X$ def. retract $\Rightarrow A \simeq X$
$i: A \to X$ inclusion map, $r: X \to A$ retract map.
$\Rightarrow \begin{cases} ri = id_A & (r(a) = a) \\ ir \sim_h id_X & (ir = r \text{ as } X \to X \sim_h id_X) \end{cases}$

Def. pt : top. sp. with a single element

X is <u>contractible</u> if $X \simeq pt$

Example $D^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1^2 + \dots + x_k^2 \le 1\}$
closed unit ball of $\mathbb{R}^k$
$\rightsquigarrow A = \{(0, \dots, 0)\}$ is a def. retr.