Chapter 0, Cell complexes

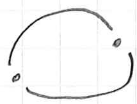
Want: present spaces as glueing of simpler parts.

- indivual pieces: copios the n-disk D^n

$$D^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^2 : x_1^2 + \dots + x_n^2 \leq 1 \}$$

- glueing happens at the "boundary"
- build up from lower dimensions

Examples

S^1 as , S^2 as 

Def. a cell complex (or CW complex) is a topological space X with following additional structures:

- subspaces $X^0 \subset X^1 \subset \cdots$ of X; X^n: n-skeleton
- X^0 is discrete (for the induced top.)
- X^n is given by glueing boundary of copies of D^n to X^{n-1}

More formally:

- index set I_n
- for each $\alpha \in I_n$, cont. map $\varphi_\alpha : S^{n-1} \to X^{n-1}$

such that X^n is homeomorphic to the quot. space

$$\left(X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} D^n \right) \right) / \quad \forall x \in S^{n-1} = \partial D^n$$

"x at the α-th comp" $\sim \varphi_\alpha(x)$

so $X^n = X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} \mathring{D}^n \right)$ as sets, with quot. top.

interior of D^n

- the topology of X is determined by those on X^n

$X = \bigcup_{n=0}^{\infty} X^n$, $U \subset X$ open iff $U \cap X_n$ open in X_n

(X has the weak top. of $\bigcup_{n=0}^{\infty} X^n$)

Notation $e^n_\alpha \subset X^n$ image of $\mathring{D}^n$ at the α-th position

n-cell of X.

Example $X = S^2$

$X^0 = S^0$ two points
$X^1 = S^1$
$X^2 = S^2 = X^3 = X^4 = \cdots$
$I_3 = \emptyset, \cdots$

Def a cell-cplx X is <u>finite dimensional</u> if

$X = X^n$ for some n (hence $X^m = X$ $m > n$)

Alternate presentation (Prop. A.2)

$X^0 \subset X^1 \subset \cdots$ s.t. $X = \bigcup_{n=0}^{\infty} X^n$, I_n ind sets

- for each $\alpha \in I_n$: $\Phi_\alpha : D^n \to X^n$ s.t.

$\Phi_\alpha|_{\mathring{D}^n} : \mathring{D}^n \to \Phi_\alpha(\mathring{D}^n)$ is homeo.

- $X^n = \coprod_{k=0}^{n} \left(\coprod_{\beta \in I_k} \Phi_\beta(\mathring{D}^k) \right)$ as sets
- $\Phi_\alpha(S^{n-1})$ is contained in $\Phi_{\beta_1}(D^{k_1}) \cup \cdots \cup \Phi_{\beta_m}(D^{k_m})$

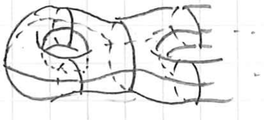
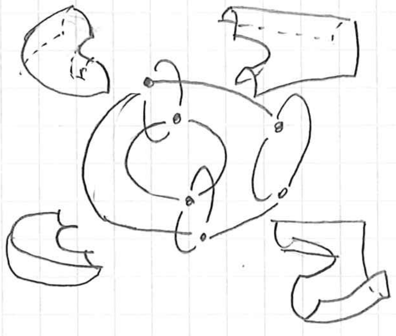
for some $k_1, \ldots, k_m < n$, $\beta_i \in I_{k_i}$

- $C \subset X$ is closed iff $\forall n, \alpha \in I_n$: $C \cap \Phi_\alpha(D^n) \subset X^n$ is closed.

Examples closed orientable surfaces.

Σ_g: 

1. easier but inefficient model.


$\rightsquigarrow$


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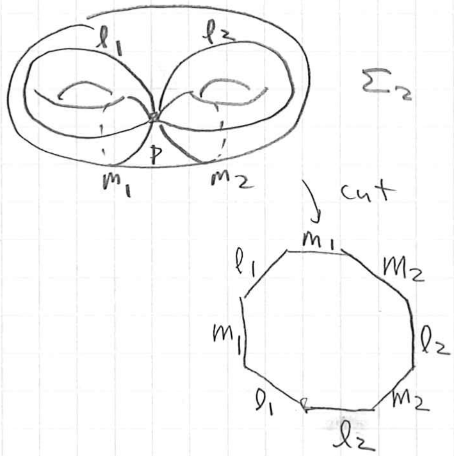
2. efficient model. : fix $p \in \Sigma_g$, take two loops for each "hole" (meridian / longitude)
$\underbrace{}_{\text{starting from } p}$
cut Σ_g along these loops
$\rightsquigarrow$ get a copy of D^2.



$X^0 = \{p\}$, $X^1 =$ union of these loops

$X^2 = \Sigma_g$.

Example S^n one 0-cell and one n-cell

$X^0 = e^0$ (singleton), $\varphi : S^{n-1} \to e^0$ $\rightsquigarrow$ $S^n \underset{\text{homeo}}{\cong} D^n / S^{n-1}$

"Subspace as a complex"

Def. X : cell complex ; $X = \coprod_{n=0}^{\infty} \left(\coprod_{\alpha \in I_n} \Phi_\alpha (\mathring{D}^n) \right)$.

a <u>subcomplex</u> of X is a closed subset $A \subset X$ which is a union of cells of X

$A = \coprod_{n=0}^{\infty} \left(\coprod_{\alpha \in J_n} \Phi_\alpha (\mathring{D}^n) \right)$ for some $J_n \subset I_n$.

we say (X, A) is a CW pair.

Rem. this depends on the cell complex structure

$S^{n-1} \subset S^n$ is a subcomplex if we take

$X^0 = S^0 \subset X^1 = S^1 \subset \cdots \subset X^n = S^n$

but not if we take $X^0 = \{*\} = X^1 = \cdots = X^{n-1} \subset X^n = S^n$