2024-1-31 no. 1

Operations on Spaces.

1. Product of cell complexes

X from I_n, $\varphi_\alpha : S^{n-1} \to X^{n-1}$ for $\alpha \in I_n$, ...

Y from J_n, $\psi_\beta : S^{n-1} \to Y^{n-1}$ for $\beta \in J_n$, ...

$\rightsquigarrow$ $X \times Y$ becomes a cell complex

n-cells : $e^m_\alpha \times e^{n-m}_\beta$ for $\alpha \in I_m$, $\beta \in J_{n-m}$

(use $\underbrace{\mathring{D}^m}_{\text{homeo to } (0,1)^m} \times \underbrace{\mathring{D}^{n-m}}_{\text{hom } (0,1)^{n-m}} \underset{\text{homeo}}{\cong} \underbrace{\mathring{D}^n}_{\text{hom } (0,1)^n}$)

2. Quotient by a subcomplex $(X, A) \rightsquigarrow X/A$.

"collapse A to a single point"

as a set : $X/A = (X \setminus A) \amalg \underbrace{\{*\}}_{\text{new point}}$

as a top. sp. : quotient of X by the equiv. rel $\sim$: $x \sim y \iff x = y$ or $x, y \in A$

X $\rightsquigarrow$ X/A

0-cells of X/A : 0-cells of $X \setminus A$ and $*$

n-cells of X/A : n-cells of $X \setminus A$.
$n > 0$

- Cone over a cell complex (or over a top. sp.)

  $CX = (X \times I)/(X \times \{0\})$ (also write $\mathrm{Con}(X)$, etc.)

  - treat I as a cell complex with 0-cells $\{0\}, \{1\}$
    one 1-cell $(0,1)$.
  - $X \times \{0\} \subset X \times I$ subcomplex:
    $\left(\bigcup_{n,\alpha} e^n_\alpha \times \{0\} \right)$ ← cells: $e^n_\alpha \times \{0\}$, $e^n_\alpha \times \{1\}$, $e^n_\alpha \times (0,1)$; $e^n_\alpha \subset X$ cell.

Prop. Write $* = [X \times \{0\}] \in CX$. Then $A = \{*\} \subset CX$ is a deform. retract

i.e. $\exists r: CX \to A$ (unique for this) $\rightsquigarrow i r \simeq \mathrm{id}_{CX}$
i: incl. $A \to CX$.

(CX is contractible)

Proof. Want: $F: CX \times I \to CX$ homotop. betw. ir and id_{CX}

at "time" $t \in I$

Formally $F(\underbrace{x, s}_{\text{coord for } X \times I}, t) = [x, st]$ first def'ed as $X \times I \times I \to CX$
take img in CX.

Step 1 this is well-def'ed as $CX \times I \to CX$.
(x,s,t) and (x',s',t') represent same points in $CX \times I$

$\Leftrightarrow x = x', s = s', t = t'$ or $s = 0 = s', t = t'$

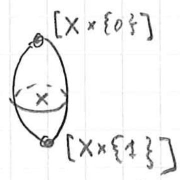
in the second case $[x, st] = [x, 0] = [x', s't']$

Step 2 $F(x,s,0) = ir([x,s])$

Step 3 $F(x,s,1) = \mathrm{id}_{CX}([x,s])$ $\square$

- Suspension

  $SX = CX/(X \times \{1\})$



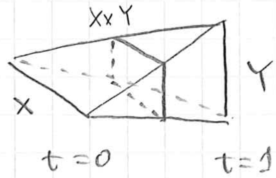
Example $SS^n = S^{n+1}$

Rem. SX is <u>not</u> contractible

- Join of two cell complexes (or spaces)

$X * Y$ = quotient of $X \times Y \times I$ by the rel $\sim$

1. $(x, y, 0) \sim (x, y', 0)$ for any $x \in X$, $y, y' \in Y$
2. $(x, y, 1) \sim (x', y, 1)$ for any $x, x' \in X$, $y \in Y$



cells : $e^n_\alpha \times \{[Y]\} \times \{0\}$, $e^n_\alpha \times e^m_\beta \times (0,1)$, $\{[X]\} \times e^m_\beta \times \{1\}$; $e^n_\alpha \subset X$, $e^m_\beta \subset Y$

Exercise when $X = Y = I$, $Z = X * Y$, identify
- k-skeleton Z^k for $k = 0, 1, 2, 3$
- glueing maps $S^k \to Z^{k-1}$

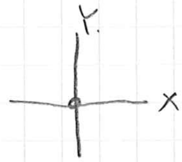
(up to $S^1 \underset{\text{homeo}}{\cong}$  ∂I^2. etc.)

- Wedge sum of two pointed cell complexes / spaces

Def: pointed space : (X, x_0) X top. sp., $x_0 \in X$.
for cell complex : assume x_0 is a 0-cell.

$X \vee Y$ = quotient of $X \amalg Y$ (Disjoint union) by the extra rel. $x_0 \sim y_0$.

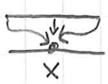
Rem. $X \vee Y \underset{\text{homeo}}{\cong} (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y$



- Smash product of pointed cell cplxes / spaces

$X \wedge Y$ = quot. of $X \times Y$ by the extra rel.

1. $(x, y_0) \sim (x', y_0)$ $x, x' \in X$
2. $(x_0, y) \sim (x_0, y')$ $y, y' \in Y$

Examples $S^0 \wedge X \cong X$  , $S^1 \wedge X \underset{\text{homeo}}{\cong} SX / (\{x_0\} \times I) \underset{\text{homotop}}{\simeq} SX$