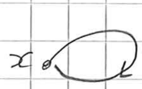
2023-2-7 no. 1

Chapter 1 Fundamental group

Basic idea : pointed sp. (X, x) $\rightsquigarrow$

loops in X based at x ; 

up to (basept-fixing) homotopy

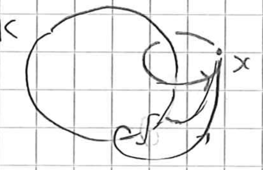
<u>group law</u> by composition of loops.

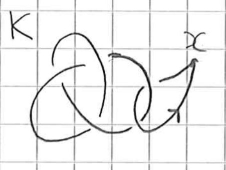
Example : <u>knot group</u>

$K \subset \mathbb{R}^3$ a knot (subspace homeo. to S^1)

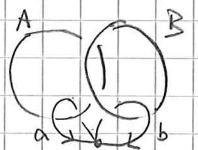
$\rightsquigarrow$ $X = \mathbb{R}^3 \setminus K$, $x \in X$ any pt

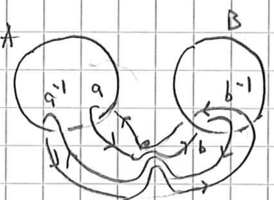
$\rightsquigarrow$ loops in X based at x : loops going around K, based at x.

<u>unknot</u> 

<u>trefoil</u> 

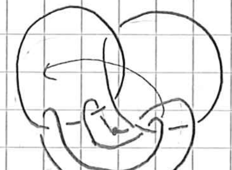
analogue for links $L \subset \mathbb{R}^3$ (subsp. homeo. to $S^1 \amalg \cdots \amalg S^1$)

<u>unlink</u> 

$\rightsquigarrow$ 

$a b a^{-1} b^{-1}$ nontriv.

<u>Hopf link</u> 

$\rightsquigarrow$ 

$a b a^{-1} b^{-1}$ homotopic to a const. loop.

Warning : "<u>link group</u>" is a quotient of $\pi_1(\mathbb{R}^3 \setminus L, *)$ with extra rels

§1.1. Quick review on paths and homotopy

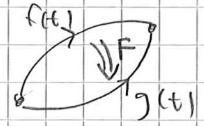
- a <u>path</u> in X is given by a cont. map $I \to X$
- we consider homotopy <u>fixing endpoints</u> :

$f \simeq g \iff f(0) = g(0),\ f(1) = g(1)$

$\exists F : \underset{\substack{\uparrow \\ \text{path param}}}{I} \times \underset{\substack{\uparrow \\ \text{deform param}}}{I} \to X,\ F(t,0) = f(t),\ F(t,1) = g(t),$

$F(0,t) = f(0),\ F(1,t) = f(1)$

cont.

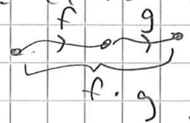


$f_s(t) = F(t,s)$ interpolating f and g

- $x, y \in X$ ⇒ path homotopy is an equiv. rel. on $\{ f : I \to X : f(0) = x, f(1) = y \}$.

write $[f]$ for the equiv. class of f.

<u>Composition</u> of paths

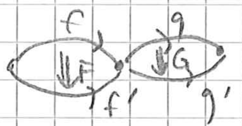


when $f, g : I \to X$, $f(1) = g(0)$ define $f \cdot g$ by

$$(f \cdot g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

- $[f \cdot g]$ only depends on $[f]$ and $[g]$

i.e. $f \simeq f'$, $g \simeq g'$ ⇒ $f \cdot g \simeq f' \cdot g'$



$$H(t,s) = \begin{cases} F(2t, s) & t \le \frac{1}{2} \\ G(2t-1, s) & \frac{1}{2} \le t. \end{cases}$$

gives $f \cdot g \simeq f' \cdot g'$.

⇝ write $[f \cdot g] = [f] \cdot [g]$

<u>Inverse</u> and <u>unit</u> paths

given $f : I \to X$, define $\tilde{f} : I \to X$ by

$\tilde{f}(t) = f(1-t)$ so $\tilde{f}(1) = f(0)$, $\tilde{f}(0) = f(1)$

- $f \cdot \tilde{f}$ path-homotopic to the const. path at $f(0)$
- $\tilde{f} \cdot f$ 〃 at $f(1)$

i.e. $[f] \cdot [\tilde{f}] = [e_x]$, $[\tilde{f}] \cdot [f] = [e_y]$ for $x = f(0)$, $y = f(1)$

const path

- $[f] \cdot [e_y] = [f] = [e_x] \cdot [f]$;

⇝ Const paths behave as "units" for path composition

Fundamental group(oid).

- $\pi_1(X) = \{[f] : f : I \to X\}$ set of path-homotop classes of paths in X, with compos.

  $\rightsquigarrow$ groupoid with base X

- $x \in X \rightsquigarrow \pi_1(X, x) = \{[f] : f : I \to X, f(0) = f(1) = x\}$

  path-homotop classes of loops based at x.

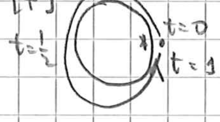
  with compos. $\rightsquigarrow$ becomes a group with unit $[e_x]$; inverse $[f]^{-1} = [\tilde{f}]$

- $\exists$ path in X from x to y $\Rightarrow \pi_1(X, x) \cong \pi_1(X, y)$

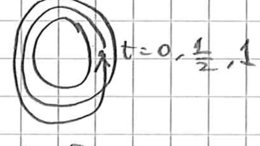
  So the precise choice of $x \in X$ is not important if X is path-connected

Examples 1. $\pi_1(S^1, *) \cong \mathbb{Z}$ conting the number of times loops go around (in counterclockwise dir.)

$[f]$



$[f]^2 = [f \cdot f]$



$[f]^{-1} = [\tilde{f}]$

Corr. to $1 \in \mathbb{Z}$ $\quad$ $2 \in \mathbb{Z}$ $\quad$ $-1 \in \mathbb{Z}$.

2. $\pi_1(S^2, *) \cong \{0\}$.

3. $\pi_1(\underbrace{\mathbb{R}^3 \setminus (\text{Hopf link})}_{\text{homotopic to } S^1 \times S^1}, *) \cong \mathbb{Z}^2$