Chap. 1. Fundamental group of circle (via covering space).

Review of $\pi_1(S^1, *) \cong \mathbb{Z}$.

look at the covering map $p : \mathbb{R} \to S^1$,

$t \mapsto (\cos 2\pi t, \sin 2\pi t)$

$0 \mapsto (1, 0) = *$

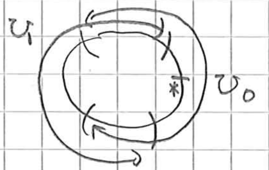
- any loop f in S^1 based at $*$ lifts to a path $\tilde{f} : I \to \mathbb{R}$ starting at 0, ending at $n \in \mathbb{Z}$ $(p(n) = *)$, $p \circ \tilde{f} = f$
- any paths $\tilde{f}, \tilde{g}$ s.t. $\tilde{f}(0) = 0 = \tilde{g}(0)$, $\tilde{f}(1) = n = \tilde{g}(n)$ are homotopic (as paths)

$\Rightarrow$ get a corresp. $\pi_1(S^1, *) \to \mathbb{Z}$, $f \mapsto \tilde{f}(1)$

Def. $p : Y \to X$ cont. map

- $U \subset X$ (open) is <u>evenly covered</u> by p if $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$ for some open sets $V_\alpha \subset Y$ s.t. $\forall \alpha \in I$ $p|_{V_\alpha} : V_\alpha \to U$ is a homeo.
- $p : Y \to X$ is a <u>covering space</u> if $\exists (U_\beta)_{\beta \in J}$ open sets in X s.t. $X = \bigcup_{\beta \in J} U_\beta$, U_β is evenly covered by p

Ex. $p : \mathbb{R} \to S^1$



$p^{-1}(U_0) = \coprod_{k \in \mathbb{Z}} (k - \frac{2\pi}{3}, k + \frac{2\pi}{3})$

$p^{-1}(U_1) = \coprod_{k \in \mathbb{Z}} (k - \frac{\pi}{3}, k + \frac{5\pi}{3})$

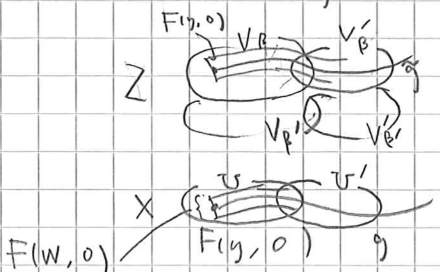
Key Thm. $p : Z \to X$ cov.

$\forall F : Y \times I \to X$, $f : Y \to Z$ s.t. $p(f(y)) = F(y, 0)$

$\exists! \tilde{F} : Y \times I \to Z$ s.t. $p(\tilde{F}(y, t)) = F(y, t)$, $\tilde{F}(y, 0) = f(y)$

Idea : for each $y \in Y$, take the path $g_y : I \to X,\ t \mapsto F(y,t)$

1. unique lift $\tilde{g}_y : I \to$ s.t. $\tilde{g}_y(0) = f(y)$
2. $\exists$ neighborhood $W \ni y$ s.t. $(y', t) \mapsto \tilde{g}_{y'}(t)$ is cont. on $W \times J$



1. find unique lift of $g_y(t)$ for small t s.t. $g_y(t) \in$ evenly covered U. then continue in next evenly covered U'
2. $\exists$(cont.) extension of $y' \mapsto g_{y'}(t)$ to a neigh. of y $\rightsquigarrow$ some neigh. W works for all t by compactness

3. Then $\tilde{F}(y,t) = \tilde{g}_y(t)$ is cont.

Corollaries $p : Z \to X$ covering

a) $f : I \to X$ starting at x_0, $\tilde{x}_0 \in p^{-1}(x_0)$

$\Rightarrow \exists!\ \tilde{f} : I \to Z$ lift of f s.t. $\tilde{f}(0) = \tilde{x}_0$

b) $F : I \times I \to X$ $F(t,0) = x_0$, $\tilde{x}_0 \in p^{-1}(x_0)$

homotop. of paths starting at x_0

$\Rightarrow \exists!\ \tilde{F} : I \times I \to Z$ lift of F s.t. $\tilde{F}(t,0) = \tilde{x}_0$

Diagramatically :

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "y \mapsto (y,0)"'] & Z \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists! \tilde{F}"] & X \end{tikzcd}$$

homotopy lifting property of covering maps.

fibrations

Combining a & b, we get a well-defined (injective) map

$$A = \{[f] : f : I \to X,\ f(0) = x_0\} \longrightarrow \{[f'] : f' : I \to Z,\ f'(0) = \tilde{x}_0\} = B$$

$$[f] \longmapsto [\tilde{f}]$$

If Z is simply connected. ($\pi_1(Z, z) \cong \{0\}$ for all z)

then $B \to p^{-1}(x_0)$, $[f'] \mapsto f'(1)$ is injective.

Induced homomorphisms.

$\varphi : (X, x_0) \to (Y, y_0)$ (cont.) map of pointed spaces

i.e. $\varphi : X \to Y$ s.t. $\varphi(x_0) = y_0$

$\rightsquigarrow \varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $[f] \to [\varphi f]$

Functoriality

- $(id_X)_* = id_{\pi_1(X, x_0)}$
- $\psi_* \varphi_* = (\psi\varphi)_*$ for $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$

Lem 1.15 (X, x_0) pt'ed sp. $(A_\alpha)_{\alpha \in I}$ open cover of X s.t.

- $x_0 \in A_\alpha$ for all α
- $A_\alpha \cap A_\beta$ path-conn. for all α, β.

Then $\forall [f] \in \pi_1(X, x_0)$ $\exists f_1, \dots, f_m$ loops at x_0

s.t. $\forall i$ $\exists \alpha_i$ s.t. $f_i : I \to A_{\alpha_i}$, $[f] = [f_1] \cdots [f_m]$

Proof. Step 1 find $s_0 = 0 < s_1 < \cdots < s_m = 1$ s.t.

$f([s_{i-1}, s_i])$ is contained in some A_{α_i}

each $t \in I$ has $\varepsilon > 0$ s.t. $f([t-\varepsilon, t+\varepsilon]) \subset A_{\alpha_t}$

for some α_t $\rightsquigarrow$ use compactness to find $(s_i)_i$

Step 2 find $f_1, \dots, f_m$.

$f(s_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$ $\rightsquigarrow$ $\exists$ path g_i (in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$) from x_0 to $f(s_i)$

$$f_1 = \underbrace{f|_{[s_0, s_1]} \cdot \tilde{g}_1}_{\text{loop in } A_{\alpha_1}}, \quad f_2 = \underbrace{\underbrace{g_1}_{\text{in } A_{\alpha_1} \cap A_{\alpha_2}} \cdot \underbrace{f|_{[s_1, s_2]}}_{\text{in } A_{\alpha_2}} \cdot \underbrace{\tilde{g}_2}_{\text{in } A_{\alpha_2} \cap A_{\alpha_3}}}_{\text{loop in } A_{\alpha_2}}, \dots$$

Then $[f_1] \cdots [f_m] = [f|_{[s_0, s_1]} \cdot \tilde{g}_1 \cdot g_1 \cdot f|_{[s_1, s_2]} \cdots]$

$= [f|_{[s_0, s_1]} \cdot f|_{[s_1, s_2]} \cdots f|_{[s_{m-1}, s_m]}]$

Cor. (Prop 1.14) $\pi_1(S^n, *) \cong \{0\}$ for $n \geq 2$.

$S^n = A_{\alpha_1} \cup A_{\alpha_2}$ $\quad A_{\alpha_1} \cong \mathring{D}^n \cong A_{\alpha_2}$, $A_{\alpha_1} \cap A_{\alpha_2} \cong S^{n-1} \times \mathring{D}^1$