(§1.1), Induced homomorphisms, continued.

Recall: $\varphi : (X, x) \to (Y, y)$ map of pointed spaces $\rightsquigarrow$ hom $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$

$[f] \mapsto [\varphi f]$

s.t. $\psi_* \varphi_* = (\psi\varphi)_*$ for $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$

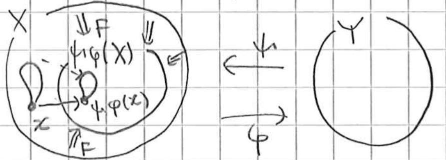
$(id_X)_* = id_{\pi_1 \cdots}$

Prop 1.18 $\varphi : X \to Y$ homotopy equiv.

$x \in X$, $y = \varphi(y)$

then $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isom.

Proof. Take $\psi : Y \to X$ s.t. $\psi\varphi \underset{\text{homotop}}{\simeq} id_X$, $\varphi\psi \simeq id_Y$



$F(x, 0) = x$, $F(x, 1) = \psi\varphi(x)$

Let $F : X \times I \to X$ be a homotop. betw. $\psi\varphi$ & id_X

$f : I \to X$, $t \mapsto F(x, t)$ "trajectory of x"

$\rightsquigarrow \pi_1(X, \psi\varphi(x)) \underset{Ad_f}{\xrightarrow{\sim}} \pi_1(X, x)$, $[g] \mapsto [f \cdot g \cdot \tilde{f}]$

Step 1 $Ad_f(\psi\varphi)_* = id_{\pi_1(X, x)}$

i.e. $f \cdot \psi\varphi g \cdot \tilde{f} \simeq g$ for loops g at x.

Put $g_t(s) = F(g(s), t)$

Then $h_t = f|_{[0,t]} \cdot g_t \cdot \widetilde{f|_{[0,t]}}$ gives a homotop.

between g and $f \cdot \psi\varphi g \cdot \tilde{f}$

Step 2 φ_* is injective.

$Ad_f(\psi\varphi)_* = Ad_f \psi_* \varphi_*$ is inj.

Step 3 ψ_* is surjective ($\Rightarrow$ so is φ by symm.)

$A\ell_{\tilde{f}} : \pi_1(X, x) \to \pi_1(X, \psi\varphi(x))$ is the inverse of $A\ell_f$

$\Rightarrow (\psi\varphi)_* = A\ell_{\tilde{f}}$: this is isom. $\Rightarrow$ surj.

$\overset{\shortparallel}{\psi_* \varphi_*}$

$\boxed{}$

Application of functoriality : topological groups

Def. a topological group is given by

a set G that is $\begin{cases} \text{a top. sp} \\ \text{a group} \end{cases}$

s.t. $m : G \times G \to G$ prod. map, $G \to G$, $g \mapsto g^{-1}$

are continuous.

Examples. $S^1 \subset \mathbb{C}$ by mult.,

matrix groups $GL_n(\mathbb{R})$, $U(n)$, ...

Thm. G top grp $\Rightarrow \pi_1(G, e)$ is commutative.

Idea : prod. map of $\Gamma = \pi_1(G, e)$ is a

grp hom from $\Gamma \times \Gamma$ to Γ.

Step 1 : $x, y \in \Gamma \rightsquigarrow (x, y) \overset{m_\Gamma}{\longmapsto} x \cdot y$

$(x,y) = (x, e_\Gamma) \cdot (e_\Gamma, y) = (e_\Gamma, y) \cdot (x, e_\Gamma)$ in $\Gamma \times \Gamma$

$\mapsto m_\Gamma(e_\Gamma, y) \cdot m_\Gamma(x, e_\Gamma) = y \cdot x$

m_Γ is hom

Step 2. m_Γ is a hom.

$m_G : G \times G \to G$ gives a retract for

$i_1 : G \to G \times G$, $g \mapsto (g, e)$

$\Rightarrow (m_G)_* (i_1)_* ([f]) = [f]$.

$(m_G)_* ([f_1], [f_2]) = (m_G)_* ((i_1)_* ([f_1]) \cdot (i_2)_* ([f_2]))$

hom.

2023-2-21 no. 3

cont. $(m_G)_*(i_1)_*([f_1]) \cdot (m_G)_*(i_2)_*([f_2]) = [f_1][f_2]$

$= m_\Pi([f_1],[f_2])$ ☒

Spaces with non comm. π_1:

 ,  , ... cannot be groups

Section 1.2 Van Kampen's th'm.

Suppose $X = A \cup B$ for some subspaces A, B

Q. fix $x \in A \cap B$ ⇝ how do we compute $\pi_1(X,x)$ from $\pi_1(A,x)$ and $\pi_1(B,x)$?

Obs: if $A \cap B$ is path-connected, any $[f] \in \pi_1(X,x)$ can be written as $[f_1] \cdots [f_m]$, s.t. f_i is a loop in A or B (based at x; say "loop in (A,x)", etc.)

formally $[f] = i_*([f_1]) \cdots i_*([f_m])$ for $i : A, B \to X$ incl.

"easy" case : $A \cap B \underset{\text{homotop.}}{\simeq} \{x\}$ ⇝ no (nontriv.) relation between $\pi_1(A,x)$ and $\pi_1(B,x)$ in $\pi_1(X,x)$

(two pres. $[f] = [f_1] \cdots [f_m] = [f'_1] \cdots [f'_n]$
f_i, f'_j loop in (A,x) or (B,x)
are rel. by trans. of the form $[f_i][f_{i+1}] ⇝ [f_i \cdot f_{i+1}]$
when $f_i, f_{i+1} \subset A$ or $f_i, f_{i+1} \subset B$

general case : $\pi_1(A \cap B, x)$ nontrivial.

⇝ any loop f in $(A \cap B, x)$ gives elements $g = [f] \in \pi_1(A,x)$ and $g' \in [f] \in \pi_1(B,x)$ that should be identified in $\pi_1(X,x)$