§1.2 Van Kampen's th'm, cont'd.

Free product of groups : make sense of a grp G with subgrps $G_1, G_2, \dots$ s.t.

- G_i's generate G.
- no rel. across different G_i's

In particular, any elem $g \in G$ should be written uniquely as $g = g_1 g_2 \cdots g_m$, $g_i \in G_{j_i}$, $j_i \neq j_{i+1}$.

Def. $(G_\alpha)_{\alpha \in I}$ collection of groups

- a <u>word</u> (for this coll.) is an expression $g_1 g_2 \cdots g_m$ with $g_i \in G_{\alpha_i}$ for some α_i, m.

  if $G_\alpha = G_\beta$ for some $\alpha, \beta \in I$, distinguish the case $\alpha_1 = \alpha$ and $\alpha_1 = \beta$, etc.
  formally consider $\left(\coprod_{\alpha \in I} G_\alpha\right)^m$.

- a <u>reduced</u> word is a word $g_1 \cdots g_m$ $(g_i \in G_{\alpha_i})$ s.t. $g_i \neq e$, $\alpha_i \neq \alpha_{i+1}$ for all i

  [allow the "empty word" $m = 0$

- the <u>free product</u> of $(G_\alpha)_\alpha$ is

$$*(G_\alpha)_{\alpha \in I} = \mathop{*}_{\alpha \in I} G_\alpha = \{ \text{reduced words } g_1 \cdots g_m : m = 0, 1, \dots \}$$

group structure : join words

$$(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$$

if g_m and h_1 belong to same G_α

compute $k = g_m h_1$ and take

$g_1 \cdots g_{m-1} k h_2 \cdots h_n$ $(k \neq e)$

$g_1 \cdots g_{m-1} h_2 \cdots h_n$ $(k = e)$ $\rightsquigarrow$ continue reduction neutral elem : empty word

inverse of $g_1 \cdots g_m$: $g_m^{-1} \cdots g_1^{-1}$

Examples 1. the free group with 2 generators.

$G_1 = G_2 = \mathbb{Z} \rightsquigarrow \mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$.

$= \{ a^{e_1} b^{f_1} \cdots a^{e_m} b^{f_m} : e_i, f_i \in \mathbb{Z}$
$e_i \neq 0$ for $i \neq 1$,
$f_j \neq 0$ for $j \neq m$ $\}$

$a \leftrightarrow 1 \in \mathbb{Z}$ for $G_1 = \mathbb{Z}$, $b \leftrightarrow 1 \in \mathbb{Z}$ for $G_2 = \mathbb{Z}$

2. $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \to \{ (n, g) : n \in \mathbb{Z}, g \in \mathbb{Z}_2 \}$
$\hookrightarrow \langle a, b : a^2 = e = b^2 \rangle$

$a \leftrightarrow (1, g)$, $b \leftrightarrow (0, g)$ ($g \in \mathbb{Z}_2$ nontriv.)

$(1,g)^2 = (1 + (-1), g^2) = (0, e) = (0, g)^2$

Variation : amalgamated free product

$(G_\alpha)_\alpha$ coll. of grps, A grp, $\varphi_\alpha : A \to G_\alpha$ hom

$\rightsquigarrow$ the amalgam free prod is

$$\underset{A}{*} (G_\alpha)_{\alpha \in I} = \left(\underset{\alpha \in I}{*} G_\alpha \right) / N$$

N : normal subgroup of $\underset{\alpha \in I}{*} G_\alpha$ generated by

$\varphi_\alpha(a) \varphi_\beta(a)^{-1}$ $a \in A$, $\alpha, \beta \in I$.

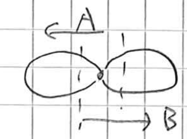
i.e. impose the rel. $a_{\text{at } \alpha} = a_{\text{at } \beta}$ on $\underset{\alpha \in I}{*} G_\alpha$

V-K thm, "usual version" $X = A \cup B$, A, B open sets

$A \cap B$ path-conn., $x \in A \cap B$

$\rightsquigarrow \pi_1(X, x) = \pi_1(A, x) \underset{\pi_1(A \cap B, x)}{*} \pi_1(B, x)$ for $i_* : \pi_1(A \cap B) \to \pi_1(A)$ etc.

Example $X = S^1 \vee S^1 = \infty \rightsquigarrow \pi_1(X, *) \cong F_2$.



A : (small) open neigh. of first copy of S^1

B : ~ second ~

$\Rightarrow A \cap B$: contractible neighborhood of $* \in A \cap B$

so $\pi_1(A, *) \cong \mathbb{Z} \cong \pi_1(B, *)$, $\pi_1(A \cap B, *) \cong \{0\}$

$\Rightarrow \pi_1(X, *) \cong \mathbb{Z} *_{\{0\}} \mathbb{Z} = \mathbb{Z} * \mathbb{Z}$

Generally : wedge sum $X \vee Y$ for pointed spaces.

assume : base pts $x \in X$, $y \in Y$ have contr. open neighborhoods U, V.

$\rightsquigarrow * \in X \vee Y$ (img of x, y) have open neighs.

$A = X \vee V$, $B = U \vee Y$. so $A \cap B = U \vee V$ contr.

then $\pi_1(X \vee Y, *) \cong \pi_1(X, x) * \pi_1(Y, y)$

$\because \pi_1(A, *) \cong \pi_1(X, x)$ bc. $X \to A$ has def. retr.

General form of the vK thm

$(A_\alpha)_{\alpha \in I}$ coll. of open sets of X s.t.

- $A_\alpha \cap A_\beta \cap A_\gamma$ path-conn. for all $\alpha, \beta, \gamma \in I$.
- $\bigcap_{\alpha \in I} A_\alpha$ contains a pt. x.

Then $\pi_1(X, x)$ is the quot. of $*_{\alpha \in I} \pi_1(A_\alpha, x)$

by the normal subgrp generated by

$(i_{\alpha\beta})_*([f])\,(i_{\beta\alpha})_*([\tilde{f}])$ for $[f] \in \pi_1(A_\alpha \cap A_\beta, x)$

$i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$ incl. map.