§1.2 Fundamental group of cell complexes

cell complex Z, skeletons Z^k $k=0,1,\cdots$

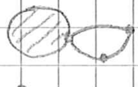
Z^0 : vertex set

Z^1 : glueing of 1-cells to vertexes

$\rightsquigarrow$ graph  $\cdots$ $\underset{\text{homotop}}{\simeq}$  $S^1 \vee \cdots \vee S^1$

fund. grp. is $\mathbb{Z} * \cdots * \mathbb{Z} \cong \mathbb{F}_k$

Z^2 : glueing of 2-cells to $Z^{(1)}$



$\rightsquigarrow$ some elems of $\pi_1(Z^{(1)}, *)$ become triv.
in $\pi_1(Z^{(2)}, *)$

$Z^3, Z^{(4)}, \cdots$: no change for $\pi_1(\sim, *)$

Prop 1 (Prop 1.26.a)

(X, x) : pointed space, I : index set

$\varphi_\alpha : S^1 \to X$ cont map, γ_α : path from x to $\varphi_\alpha(*)$
for $\alpha \in I$

$Y = (X \amalg (\coprod_{\alpha \in I} D^2)) / \varphi_\alpha(p) \sim p \in S^1 = \partial D^2$ at α-th pos

result of glueing 2-cells e^2_α to X using φ_α

$\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$: loop based at x. 

N : normal subgroup of $\pi_1(X, x)$ generated by
$[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha]$ $(\alpha \in I)$

then $\pi_1(Y, x) \cong \pi_1(X, x)/N$.

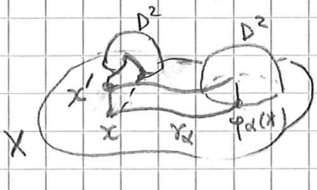
Proof : outline : find $Z \simeq Y$, $Z = A \cup B$ s.t.
$A \simeq X$, B contr., the img of $\pi_1(A \cap B, *) \to \pi_1(A, *)$
is generated by $[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha]$ $(\alpha \in I)$

$\rightsquigarrow \pi_1(Y, x) \cong \pi_1(Z, *) \cong \pi_1(A, *) \underset{\pi_1(A \cap B, *)}{*} \pi_1(B, *) \cong \pi_1(X, x)/N$

cont. $Z = (Y \amalg (\coprod_\alpha I \times I))/$ glue $I\times\{0\}$ to γ_α

$\{1\}\times I$ to D^2 at α-th pos.

identify $\{0\}\times I$ from different α.



basept of Z : x' = img of $(0,1)$

$A : Z \setminus \{\varphi_\alpha(0,0) : \alpha \in I\}$

$B : Z \setminus X$

Prop 2 (Prop 1.26.b)

similar const. as Prop 1, but with $\varphi_\alpha : S^k \to X$

($k \geq 2$ fixed) induces $\pi_1(X,x) \cong \pi_1(Y,x)$

Proof same const. of Z, A, B gives

$Z \simeq Y$, $A \simeq X$, B contr, $\pi_1(\underbrace{A\cap B}_{\text{homotop to } S^k\vee\cdots\vee S^k}, *) \cong \{e\}$ $\square$

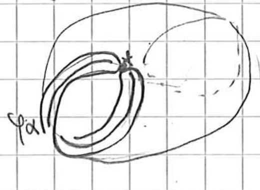
Exercise Prob. 7 $X = S^2/$ north pole ~ south pole

put a cell cplx str. on X

and compute $\pi_1(X,*)$

structure 1 : $X^0 = \{*\}$, $X^1 = S^1$, $I_2 = \{\alpha\}$

with



so $\pi_1(X^1, *) \cong \mathbb{Z}$

γ_α is const path at $*$

$[\varphi_\alpha] = [\underset{\substack{\uparrow\\ \text{coord map } S^1\to X^1}}{f}\cdot \tilde{f}] = e \in \pi_1(X^1,*)$

$\rightsquigarrow \pi_1(X,*) \cong \pi_1(X^1,*)/\{e\} \cong \mathbb{Z}$.

Structure 2 : $X^0 = \{*\}$, $X^1 = S^1\vee S^1$ ($I_1 = \{\alpha,\beta\}$)

$I_2 = \{\alpha', \beta'\}$ f_α, f_β :

$\varphi_{\alpha'} = f_\alpha \cdot \tilde{f}_\beta$, $\varphi_{\beta'} = f_\beta \cdot \tilde{f}_\alpha$.

so $\pi_1(X^1,*) \cong \mathbb{Z}*\mathbb{Z} = \langle a, b\rangle$, $\gamma_{\alpha'}, \gamma_{\beta'}$: const

img of $[\varphi_{\alpha'}] = ab^{-1}$, img of $[\varphi_{\beta'}] = ba^{-1}$

$\Rightarrow \pi_1(X,*) \cong \langle a, b \mid ab^{-1} = e\rangle \cong \mathbb{Z}$.

$a \longleftrightarrow 1$

$b \longleftrightarrow -1$

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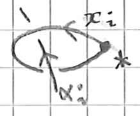
Prob 22 Wirtinger presentation of knot group

knot $K \subset \mathbb{R}^3 \rightsquigarrow$ put orientation on K, project K to a plane, label segments as $\alpha_1, \alpha_2, \cdots$

ex. 

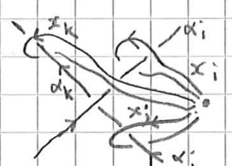
$*$: fixed basept. of $\mathbb{R}^3 \setminus K$

each α_i gives an elem. of $\pi_1(\mathbb{R}^3 \setminus K, *)$



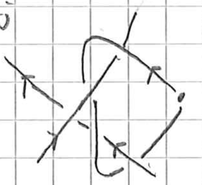
x_i : goes counterclockwise around α_i when we look at the dir. of α_i

$\rightsquigarrow$ each crossing gives a relation



take $x_k \cdot x_i$ $\rightsquigarrow$

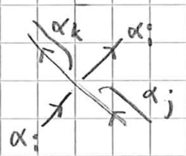
take $x_i \cdot x_j$ $\rightsquigarrow$



$\sim$

$\Rightarrow$ $x_k \cdot x_i = x_i \cdot x_j$ for each crossing

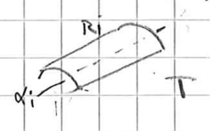
2-dim. cell complex model of $\mathbb{R}^3 \setminus K$

"look from the other side of the plane" so each crossing becomes 

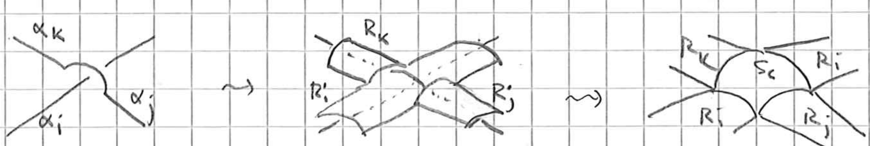
, floating over a "Table" T square

put a "tube" around each α_i

↳ rectangle R_i glued to T



over each crossing c, put a square S_c



X : result of glueing R_i, S_c, and T.

each R_i gives elem $x_i \in \pi_1(X, *)$

each S_c gives rel. $x_i x_k x_i^{-1} x_j^{-1} = e$.