§2.1 Simplicial homology cont'd.

Recall: $(X, (\sigma_\alpha : \Delta^n \to X)_{\alpha \in I_n}\ n = 0, 1, \dots)$ Δ-cplx

$\rightsquigarrow$ $\Delta_n(X) = \{ \sum_{\alpha \in I_n} n_\alpha \cdot \sigma_\alpha : n_\alpha \in \mathbb{Z}$ fin. supp.$\}$
n-chains

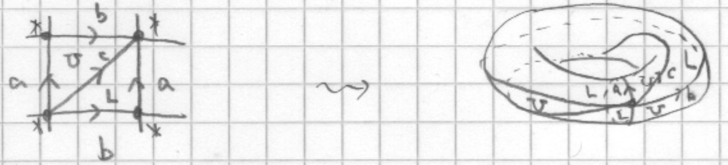
$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X),\ \sigma_\alpha \mapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

$H_n^\Delta(X) = \ker \partial_n / \operatorname{img} \partial_{n+1}$ $\quad (H_0^\Delta(X) = \Delta_0(X) / \operatorname{img} \partial_1)$

Basic idea: "$\dim H_n^\Delta(X)$" counts the number of $S^n \cong \partial \Delta^{n+1} \to X$ that are not from (the bdry of) $B^{n+1} \cong \Delta^{n+1} \to X$

Example 1 2-dim torus surf.

$X = S^1 \times S^1 \cong \mathbb{R}^2 / \mathbb{Z}^2$

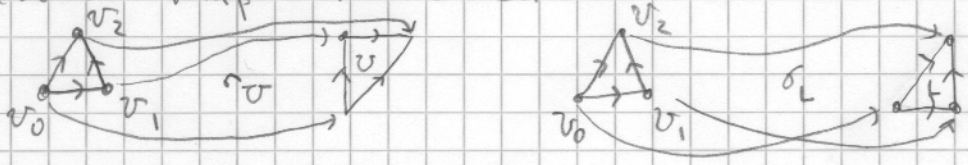


0-dim cell: σ_*

1-dim cells: $\sigma_a, \sigma_b, \sigma_c$

2-dim cells: σ_U, σ_L, (no higher dim cells)

coord map of 2-cells



$\partial_2 : \Delta_2(X) \to \Delta_1(X)$ is

$\sigma_U \mapsto \sigma_a - \sigma_b + \sigma_c,\quad \sigma_L \mapsto \sigma_a - \sigma_b + \sigma_c$

$\partial_1 : \Delta_1(X) \to \Delta_0(X)$ is

$\sigma_a \mapsto \sigma_* - \sigma_* = 0,\quad \sigma_b \mapsto 0,\quad \sigma_c \mapsto 0$

i.e. $\ker \partial_2 = \{ n \cdot \sigma_U - n \cdot \sigma_L : n \in \mathbb{Z} \}$

$\operatorname{img} \partial_2 = \{ n(\sigma_a - \sigma_b + \sigma_c) : n \in \mathbb{Z} \}$

$\ker \partial_1 = \Delta_1(X)$, $\operatorname{img} \partial_1 = 0$

$H_2^\Delta(X) = \{n(\sigma_U - \sigma_L) : n \in \mathbb{Z}\} \cong \mathbb{Z}$

$H_1^\Delta(X) = \Delta_1(X) / \{n(\sigma_a - \sigma_b + \sigma_c) : n \in \mathbb{Z}\} \cong \mathbb{Z}^2$

$[n\sigma_a + n'\sigma_b + n''\sigma_c] \mapsto (n, n')$

from $[\sigma_c] = [\sigma_b] - [\sigma_a]$.

$H_0^\Delta(X) = \Delta_0(X) \cong \mathbb{Z}$

Example 2 $X = S^n = (D^n \amalg D^n)/$ identify pts on boundaries

$D^n \cong \Delta^n \rightsquigarrow \Delta$-complex structure on S^n

with two n-dim cells σ_L, σ_U

their boundary identified.

($\rightsquigarrow (n+1)$ of $(n-1)$-dim cells, ...

$\binom{n+1}{k}$ of $(n-k)$-dim cells)

then $\sigma_U - \sigma_L$ generates $\ker \partial_n$.

($H_n^\Delta(X) \cong \mathbb{Z}$).

Connection to the Euler characteristic of polyhedra

P : polyhedron with vert set V, edge set E

face set F

$C_0 = \mathbb{Z}V = \{\sum_{v \in V} n_v \cdot v : n_v \in \mathbb{Z}$ fin supp. $\}$

$C_1 = \mathbb{Z}E$, $C_2 = \mathbb{Z}F$; fix orientations on $e \in E$, $f \in F$.

$\partial_2 : C_2 \to C_1$, $f \mapsto \sum_{e \in \partial f} (-1)^{\mathrm{sig}(e,f)} \cdot e$.

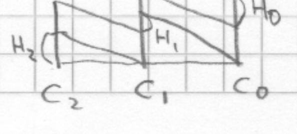
$\mathrm{sig}(e,f) = \begin{cases} 0 & \text{ori. of } e \text{ matches the induced one from } f \\ 1 & \text{otherwise} \end{cases}$

$\partial_1 : C_1 \to C_0$, $e \mapsto$ (end of e) $-$ (start of e)

$\chi(P) = |V| - |E| + |F| = \mathrm{rk}\, C_0 - \mathrm{rk}\, C_1 + \mathrm{rk}\, C_2$

this is eq. to $\mathrm{rk}\, H_0 - \mathrm{rk}\, H_1 + \mathrm{rk}\, H_2$

$H_i = \ker \partial_i / \mathrm{im}\, \partial_{i+1}$



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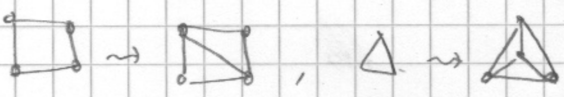
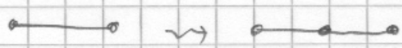
subdivision of a face 

does not change $\chi(P)$.

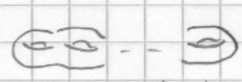
ditto for div. of an edge 

when all faces are triangles (Δ^2) we are

computing $H_i^\Delta(P)$ $(i=0,1,2)$

Then we want to show that

1. $H_i^\Delta(P)$ (in gen. $H_i^\Delta(X)$ for Δ-cplx) only depends on the homotopy type of P

2. $\sum_{i=0}^{2}(-1)^i \dim H_i^\Delta(M_g) = 2-2g$ for M_g  g holes

item 2 can be achieved with concrete models

$H_0^\Delta(M_g) \cong \mathbb{Z}$, $H_1^\Delta(M_g) \cong \mathbb{Z}^{2g}$, $H_2^\Delta(M_g) \cong \mathbb{Z}$

from 

$\rightsquigarrow M_g$: two 0-cells, $6g$ 1-cells, $4g$ 2-cells

to get item 1, we are going to

- define another model of homology

  $H_i(X)$ singular homology

  that is defd using the underlying top. sp. for X

- check $H_i^\Delta(X) \cong H_i(X)$ for Δ-cplx X.

(- and $H_i(X)$ only depends on the homotopy type of X)