§2.1. Singular homology

More universal construction of homology

- works for any top. sp. X
- functorial. $X \xrightarrow{f} Y \rightsquigarrow H_i(X) \xrightarrow{f_*} H_i(Y)$
- homotopy invar. $f, g : X \to Y$, $f \simeq g$
  $\to f_* = g_*$

(but difficult to compute $H_i(X)$ directly)

How : look at arbitrary cont maps $\Delta^n \xrightarrow{\sigma} X$
("singular" simplexes in X)
& work with their lin. combs.

Def. X : top sp., $n \in \mathbb{N}$

a <u>singular</u> <u>n-simplex</u> in X is given by a cont. map $\sigma : \Delta^n \to X$
(Δ^n : std n-simplx)

$\mathrm{Sing}_n(X) = \{ \Delta^n \xrightarrow{\sigma} X \text{ sing. n-simplx} \}$

$C_n(X) = \mathbb{Z}\,\mathrm{Sing}_n(X) = \{ \sum_{\sigma \text{ as above}} n_\sigma \cdot \sigma : n_\sigma \in \mathbb{Z}, \text{ fin. supp.} \}$

the group of <u>singular</u> <u>n-chains</u> in X

$\partial_n : C_n(X) \to C_{n-1}(X)$ hom s.t.

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]} \circ \varphi_i$$

where $\Delta^n = [v_0, \dots, v_n]$, $\varphi_i : \Delta^{n-1} \xrightarrow{\sim} [v_0, \dots, \hat{v}_i, \dots v_n]$
affine map preserving ord. of vertices.

Lem $\partial_{n-1} \circ \partial_n = 0$

Same proof as before.

Def. the n-th singular homology group of X is

$$H_n(X) = H_n^{\mathrm{sing}}(X) = \ker \partial_n / \mathrm{img}\, \partial_{n+1}$$

Caution. X infinite (as set) $\Rightarrow$ $\mathrm{Sing}_n(X)$ infinite.

$\rightsquigarrow$ no guarantee if "dim" $H_n(X) < \infty$

or $H_n(X) = 0$ for $n \gg 0$.

Basic properties of $H_n(X)$

Prop 2.6. $(X_\alpha)_{\alpha \in I}$ path components of X

$$\Rightarrow H_n(X) \cong \bigoplus_{\alpha \in I} H_n(X_\alpha).$$

Pf. idea : any $\sigma : \Delta^n \to X$ has img in some X_α

$$\Rightarrow \mathrm{Sing}_n(X) = \coprod_{\alpha \in I} \mathrm{Sing}_n(X_\alpha).$$

$$\Rightarrow C_n(X) = \bigoplus_{\alpha \in I} C_n(X_\alpha)$$

∂_n respects this decomp.

Prop 2.7 $X \neq \emptyset$, path conn. $\Rightarrow H_0(X) \cong \mathbb{Z}$.

Pf. S.0 $\mathrm{Sing}_0(X) \cong X$ as sets $(\Delta^0 = \{*\})$.

$\quad \overset{\shortparallel}{\{\Delta^0 \to X\}}$.

$\Rightarrow H_0(X) \cong \mathbb{Z}X / \mathrm{img}\, \partial_1$

S.1 $\mathrm{img}\, \partial_1 = \{ n(x-y) : x, y \in X \}$

given x, y ; take $f : \Delta^1 \to X$ path

from x to y $(\Delta^1 \cong I)$

$\Rightarrow \partial f = f|_{[v_1]} - f_{[v_0]} = y - x$

S.2 $H_0(X) \cong \mathbb{Z}$

$\left[\sum_{x \in X} n_x \cdot x \right] \mapsto \sum_x n_x$ is an iso. $H_0(X) \to \mathbb{Z}$.

$([y] = [x]$ from $[y-x] \in \mathrm{img}\, \partial_1)$ ☑

Props 2.6 & 2.7, X top $\Rightarrow H_0(X) \cong \mathbb{Z}\pi_0(X)$

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Prop 2.8 $X = \{*\} \Rightarrow H_n(X) \cong \begin{cases} \mathbb{Z} & (n=0) \\ 0 & (n>0) \end{cases}$

Pf. $\mathrm{Sing}_n(X) = \{\sigma_n\}$ for the unique $\Delta^n \xrightarrow{\sigma_n} X$.

so $C_n(X) = \mathbb{Z}\sigma_n$

$\partial_n : C_n(X) \to C_{n-1}(X)$ is $\sigma_n \mapsto \sum_{i=0}^{n} (-1)^i \sigma_{n-1}$

i.e. $\partial_n \sigma_n = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even, } n>0 \\ 0 & \text{if } n \text{ is odd} \end{cases}$

$\Rightarrow \ker \partial_n = \begin{cases} 0 & n \text{ even, } >0 \\ C_n(X) & n \text{ odd or } =0 \end{cases}$ $\quad \mathrm{img}\, \partial_{n+1} = \begin{cases} 0 & n \text{ ev.} \\ C_n(X). & n \text{ odd} \end{cases}$

$\Rightarrow H_0(X) \cong \mathbb{Z}$ (also from 2.7), $H_n(X) = 0$ $(n>0)$ $\square$

Functoriality

Given $X \xrightarrow{f} Y$ we get $H_n(X) \xrightarrow{f_*} H_n(Y)$ from:

$f_\# : \mathrm{Sing}_k(X) \to \mathrm{Sing}_k(Y)$, $\sigma \mapsto f \circ \sigma$

$\leadsto$ hom $f_* : C_n(X) \to C_n(Y)$, $\sum_\sigma n_\sigma \cdot \sigma \mapsto \sum_\sigma n_\sigma \cdot (f \circ \sigma)$

Lem $f_* \partial_n = \partial_n f_*$ as maps $C_n(X) \to C_{n-1}(Y)$

Pf. both map σ to $\sum (-1)^i (f\circ\sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

Then $f_*(\ker \underset{C_n(X)\to C_{n-1}(X)}{\partial_n}) \subset \ker \underset{C_n(Y)\to C_n(Y)}{\partial_n}$

$f_*(\mathrm{img}\, \underset{C_{n+1}(X)\to C_n(X)}{\partial_{n+1}}) \subset \mathrm{img}\, \underset{C_{n+1}(Y)\to C_n(Y)}{\partial_{n+1}}$

So we get an induced map $f_* : H_n(X) \to H_n(Y)$

$X \xrightarrow{f} Y \xrightarrow{g} Z$ gives $(gf)_* = g_* f_*$

$(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$

Next: if $f, g : X \to Y$, $f \simeq g$ then $f_* = g_*$