long exact seq. of homology

setting : short ex. seq of chain cplxes

$$0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$$

want : long ex. seq. $\xrightarrow{\partial} H_n(C_\bullet) \xrightarrow{f_*} H_n(C'_\bullet) \xrightarrow{g_*} H_n(C''_\bullet) \xrightarrow{\partial} H_{n-1}(C_\bullet) \xrightarrow{f_*} \cdots$

Key lem : "snake lemma"

given
$$\begin{array}{ccccccc} & & A_1 & \xrightarrow{i} & A_2 & \xrightarrow{j} & A_3 & \to 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & \\ 0 & \to & B_1 & \to & B_2 & \to & B_3 & \end{array}$$
ex. row, comm. squares

$\exists$ map $\partial : \ker f_3 \to \operatorname{cok} f_1$ s.t.

$$\ker f_1 \xrightarrow{i} \ker f_2 \xrightarrow{j} \ker f_3 \xrightarrow{\partial} \operatorname{cok} f_1 \xrightarrow{i_*} \operatorname{cok} f_2 \xrightarrow{j_*} \operatorname{cok} f_3$$

is exact.

Define ∂ by $\partial x = [z]$ for $z \in B_1$ s.t. $\exists y \in A_2$ $j(y) = x$, $f_2(y) = i(z)$

ambiguity of y : $w \in A_1$ $\rightsquigarrow$ changes z to $z+w$

$\rightsquigarrow [z] = [z+w]$ in $\operatorname{cok} f_1$, hence ∂ well-defi'ed.

Exactness by "diagram chasing".

$\ker i_* = \operatorname{img} \partial$: take $[z] \in \operatorname{cok} B_1$ s.t. $i_*([z]) = 0$

this means $i(z) \in \operatorname{img} f_2$ i.e. $\exists y$ $i(z) = f(y)$

then $x = j(y)$ satisfies $\partial(x) = [z]$

How to get the long ex seq. of $H_n(C_\bullet)$ etc.

Put $Z_n(C_\bullet) = \ker(\partial_n : C_n \to C_{n-1})$

$B_n(C_\bullet) = \operatorname{img}(\partial_{n+1} : C_{n+1} \to C_n)$

so $H_n(C_\bullet) = Z_n(C_\bullet) / B_n(C_\bullet)$

Take
$$\begin{array}{ccccccc} & C_n/B_n(C_\bullet) & \to & C'_n/B_n(C'_\bullet) & \to & C''_n/B_n(C''_\bullet) & \to 0 \\ & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\ 0 \to & Z_{n-1}(C_\bullet) & \to & Z_{n-1}(C'_\bullet) & \to & Z_{n-1}(C''_\bullet) & \end{array}$$

$$\begin{cases} \ker(\partial : C_n / B_n(C_\bullet) \to Z_{n-1}) \cong H_n(C_\bullet) \\ \operatorname{cok}(\partial : \sim\sim\sim) \cong H_{n-1}(C_\bullet) \end{cases}$$

$\rightsquigarrow$ get ex. seq. $H_n(C_\bullet) \to H_n(C'_\bullet) \to H_n(C''_\bullet) \xrightarrow{\partial} H_{n-1}(C_\bullet) \to \cdots$

Def. $A \subset X$ subsp $\quad C_n(X, A) = C_n(X)/C_n(A)$

$\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ induced by

$\partial_n : C_n(X) \to C_{n-1}(X)$

$H_n(X, A) = H_n(C_\bullet(X, A))$

n-th relative homology group of (X, A)

We have a long ex. seq.

$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$

Def. <u>reduced</u> <u>homology</u> $\tilde{H}_n(X)$

$\varepsilon : C_0(X) \to \mathbb{Z}$, $\sum_{x \in X} n_x \cdot x \mapsto \sum_x n_x$

$\{\text{lin comb. of } x \in X\}$

$(\varepsilon \partial_1 = 0)$, $\tilde{H}_0(X) = \ker \varepsilon / \operatorname{img} \partial_1$

$\tilde{H}_n(X) = H_n(X) \quad n > 0$

Rem. $H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$ for nonempty X.

Prop 2.22 $\quad \phi \neq A \subset X$ closed.

Suppose $\exists\, U \subset X$ open $A \subset U$, A is a def. retr. of U $\quad (\exists\, r : U \to A \quad r(a) = a$

$i \circ r \simeq id_U$)

Then $H_n(X, A) \cong \tilde{H}_n(X/A)$

Rem CW pair (X, A) satisfies this

Key thm (2.20 ; excision). $\quad Z \subset Y \subset X$ s.t. $\bar{Z} \subset \mathring{Y}$

then $H_n(X \setminus Z, Y \setminus Z) \cong H_n(X, Y)$

Th'm 2.20 ⇒ Prop 2.22

Take $X \supset U \supset A$ s.t. $A \subset U$ is def. retr.

obs. { then $i : A \to U$ is homotop. equiv.

$\Rightarrow H_n(A) \cong H_n(U) \Rightarrow H_n(U, A) = 0$

Look at the diag

$$\begin{array}{ccccc} H_n(X, A) & \to & H_n(X, U) & \xleftarrow[\text{iso by exc.}]{} & H_n(X \setminus A, U \setminus A) \\ \downarrow \text{want: iso} & & \downarrow & \swarrow & \\ H_n(X/A, A/A) & \to & H_n(X/A, U/A) & & \\ \| \wr & & & & \\ \tilde{H}_n(X/A) & & & & \end{array}$$

Step 1 $H_n(X, A) \to H_n(X, U)$ is iso

long ex seq. $\cdots \to \underset{0}{\underset{\|}{H_n(U, A)}} \to H_n(X, A) \to H_n(X, U) \to \cdots$

Step 2 $H_n(X/A, A/A) \to H_n(X/A, U/A)$ is iso

U/A contractible

Step 3 $H_n(X \setminus A, U \setminus A) \to H_n(X/A, U/A)$ is iso

$X \setminus A \cong (X/A) \setminus (A/A)$, etc. & excision ∎

Idea for Th'm 2.20

equiv. form: $A\ (= Y),\ B\ (= X \setminus Z) \subset X$ s.t.

$X = \mathring{A} \cup \mathring{B}$

Then $H_n(X, A) \cong H_n(B, A \cap B)$

C'_n = span of $\mathrm{Sing}_n(A)$ and $\mathrm{Sing}_n(B)$ in $C_n(X)$

(• $C'_\bullet \to C_\bullet(X)$ (incl.) is chain homotop equiv.

(• $C'_n / C_n(A) \cong C_n(B, A \cap B)$

$$\rightsquigarrow H_n(A) \begin{array}{l} \nearrow \\ \searrow \end{array} \begin{array}{ccc} H_n(C'_\bullet) & \to & H_n(C'_\bullet / C_\bullet(A)) \cong H_n(B, A) \\ \downarrow \wr & & \wr\| \\ H_n(X) & \to & H_n(X, A) \end{array} \begin{array}{l} \searrow^{\partial} \\ \nearrow_{\partial} \end{array} H_{n-1}(A) \to \cdots$$