§2.2 Application of homology

degree of maps $f: S^n \to S^n$

¿ how many such maps are there up to homotopy?

$n=1$; $\pi_1(S^1,*) \cong \mathbb{Z}$ each $k \in \mathbb{Z}$ corresp. to $z \mapsto z^k$ ($S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$)

to capture similar phenomena : look at $H_n(S^n) \cong \mathbb{Z}$

f induces $f_*: H_n(S^n) \to H_n(S^n)$ $\leftrightarrow$ hom $\mathbb{Z} \to \mathbb{Z}$

Def. the <u>degree</u> of f : $k \in \mathbb{Z}$ s.t.

$$f_*(x) = kx \qquad x \in H_n(S^n) \; ; \; k = \deg f$$

$f, g: S^n \to S^n$ $\quad \underset{\text{homotopic}}{f \simeq g} \Rightarrow \deg f = \deg g$
($\Leftarrow$ also).

¿ given $k \in \mathbb{Z}$, can we find $f: S^n \to S^n$ s.t. $\deg f = k$?

¿ can we localize the computation of $\deg f$?

Example $\deg f = -1$: $f(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1}, -x_n)$
($(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, $\Sigma |x_i|^2 = 1$.)

Write $S^n = U \cup L$ $\quad U = \{ (x_0, \ldots, x_n) : x_n \geq 0 \}$
$\qquad L = \{ (x_0, \ldots, x_n) : x_n \leq 0 \}$

$U \cong D^n \cong L$ $\rightsquigarrow$ treat these as n-simplexes.

$\sigma_U : \Delta^n \to U$, $\sigma_L : \Delta^n \to L$ coord maps s.t.

$\sigma_U |_{\partial \Delta^n} = \sigma_L |_{\partial \Delta^n}$ $\quad$ ($\partial \Delta^n \to \{ (x_0, \ldots, x_n) : x_n = 0 \}$)

- $\sigma_U - \sigma_L$ generates $H_n^\Delta(S^n) \cong \mathbb{Z}$.
- $f_*(\sigma_U - \sigma_L) = \sigma_L - \sigma_U$.

Localization of degree

recall $x \in S^n$, U : open neighborhood of x

$\rightsquigarrow$ n-th local homology at x $H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$

$$\overset{\cong}{H_n(S^n, S^n \setminus \{x\})}$$

fix a generator $\alpha = [S^n] \in H_n(S^n)$

$\rightsquigarrow$ generator $\alpha_x \in H_n(S^n, S^n \setminus \{x\})$ as the img of α ($\alpha_x = \bar{\alpha}$ "restr.")

Def. $x, y \in S^n$. suppose $f(x) = y$

the local degree of f at x : $\deg f|_x = k \in \mathbb{Z}$ s.t.

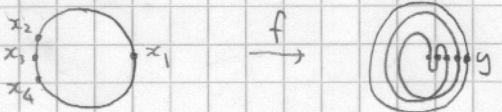
$$f_*(\alpha_x) = k \cdot \alpha_y$$

Prop 2.30 $f : S^n \to S^n$, $y \in S^n$

suppose $f^{-1}(y)$ is finite $f^{-1}(y) = \{x_1, \cdots, x_k\}$

Then $\deg f = \sum_{i=1}^{k} \deg f|_{x_i}$

Ex.



$\deg f|_{x_1} = \deg f|_{x_2} = \deg f|_{x_4} = 1$

$\deg f|_{x_3} = -1$.

Proof Idea : factorize $f_* : H_n(S^n) \to H_n(S^n)$ by

$$H_n(S^n) \to H_n(S^n, S^n \setminus \{x_1\}) \oplus \cdots \oplus H_n(S^n, S^n \setminus \{x_k\}) \to H_n(S^n)$$

$$\beta \mapsto \bar{\beta} \oplus \cdots \oplus \bar{\beta}, \quad \bar{\gamma}_1 \oplus \cdots \oplus \bar{\gamma}_k \mapsto \sum (\deg f|_{x_i}) \gamma_i$$

Step 1 $H_n(S^n, S^n \setminus \{x_1\}) \oplus \cdots \oplus H_n(S^n, S^n \setminus \{x_k\})$

$$\cong H_n(S^n, S^n \setminus \{x_1, \cdots, x_k\})$$

take small neighborhoods U_i of x_i ($U_i \cap U_j = \emptyset$)

$U = U_1 \cup \cdots \cup U_k$ neigh. of $\{x_1, \cdots, x_k\}$

excision $\Rightarrow H_n(S^n, S^n \setminus \{x_1, \cdots, x_k\}) \cong H_n(U, U \setminus \{x_1, \cdots, x_k\})$

$$\underset{U_i \cap U_j = \emptyset}{\cong} \bigoplus_{i=1}^{k} H_n(U_i, U_i \setminus \{x_i\}) \cong \bigoplus_{i=1}^{k} H_n(S^n, S^n \setminus \{x_i\})$$

Step 2. restr. $H_n(S^n) \to H_n(S^n, S^n \setminus \{x_1, \cdots, x_k\})$ is

$$H_n(S^n) \to \bigoplus H_n(S^n, S^n \setminus \{x_i\}), \quad \alpha \mapsto \bar{\alpha} \oplus \cdots \oplus \bar{\alpha}$$

Step 3 $\quad f_* : H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \to H_n(S^n, S^n \setminus \{y\})$

(well defied by $f(S^n \setminus \{x_1, \dots, x_k\}) \subset S^n \setminus \{y\}$)

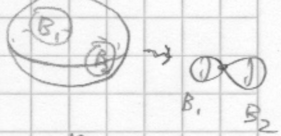
is given by $\quad \bar{\gamma}_1 \oplus \cdots \oplus \bar{\gamma}_k \mapsto \sum_{i=1}^{k} (\deg f|_{x_i}) \gamma_i$

- $H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{f_*} H_n(S^n, S^n \setminus \{y\})$ is $\bar{\gamma} \mapsto \deg f|_{x_i} \gamma$
- $H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{i_*} H_n(U, U \setminus \{x_1, \dots, x_k\})$ $\quad \underset{U_1 \cup \cdots \cup U_k}{\|}$

  is the incl. of i-th direct summand
- $H_n(U, U \setminus \{x_1, \dots, x_k\}) \cong H_n(S^n, S^n \setminus \cdots)$ $\quad \blacksquare$

Ex. $\deg f = k, -k \quad k = 2, 3, \dots$

take disj. balls $B_1, \dots, B_k \subset S^n$ $\quad (B_i \cong D^n)$



$A = S^n \setminus (\mathring{B}_1 \cup \cdots \cup \mathring{B}_k) \rightsquigarrow S^n / A \cong \underbrace{S^n \vee \cdots \vee S^n}_{k}$

$q : S^n \vee \cdots \vee S^n \to S^n$ $\quad$ id on each factor

$f = q \circ p, \quad p : S^n \to S^n / A$ $\quad$ can. surj

$y \in S^n \quad (* \neq y$ for the basept used in $S^n \vee \cdots \vee S^n)$

$\rightsquigarrow f^{-1}(y) = \{x_1, \dots, x_k\} \quad x_i \in \mathring{B}_i$

$\deg f|_{x_i} = 1 \quad (\deg p|_{x_i} = 1, \ \deg q|_{y \text{ at } i\text{-th}} = 1)$

$\rightsquigarrow \deg f = \sum_{i=1}^{k} \deg f|_{x_i} = k$

compose f with g s.t. $\deg g = -1 \Rightarrow \deg gf = -k$ $\quad \blacksquare$