§ 1.3 Covering spaces

Recall (Feb 14). <u>Covering</u> <u>space</u> of X :

$p : Y \to X$ locally looking like $V_1 \amalg V_2 \amalg \cdots$

$U \subset X$ open neigh. of $x \in X$

$V_i \subset Y$ open, homeo to U (by $p|_{V_i} : V_i \to U$)

$p^{-1}(U) = V_1 \amalg V_2 \amalg \cdots$ for the induced top

U is evenly covered by p.

Ex. $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \ (\cong S^1)$, $t \mapsto [t]$.

Blancket assumption: spaces are (path) conn.

and <u>locally</u> <u>path</u> <u>conn.</u> ($\forall x \in X$, $\underset{\text{open}}{U} \ni x$ $\exists$ $x \in V \subset U$

V is path conn)

¿ What is "the biggest covering" over X ?

$\tilde{X}$ conn. $p : \tilde{X} \to X$ cov. s.t.

any other cov. $q : Y \to X$ factors p $(\tilde{X} \to Y \to X)$

Prop 1.33 $x_0 \in X$, $y_0 \in q^{-1}(x_0)$, $\tilde{x}_0 \in p^{-1}(x_0)$

Then $\exists$ $\tilde{X} \xrightarrow{f} Y$ s.t. $q \circ f = p$ iff $p_*(\pi_1(\tilde{X}, x_0))$

$\subset q_*(\pi_1(Y, y_0))$

Then a universal $\tilde{X}$ should have the smallest $\pi_1(\tilde{X}, x_0)$

guaranteed if $\pi_1(\tilde{X}, x_0) = \{e\}$. ($\tilde{X}$ is <u>simply</u>

<u>connected</u>)

Pf. Only if : $q_* f_* = p_*$ and $f_*(\pi_1(\tilde{X}, x_0)) \subset \pi_1(Y, y_0)$

If : Step 1 Candidate for $f(\tilde{x})$

$\gamma^{\tilde{x}}$: path (in $\tilde{X}$) from $\tilde{x}_0$ to $\tilde{x}$

set $\gamma^x = p \circ \gamma^{\tilde{x}}$ path in X, γ' : lift of γ^x to

a path in Y, starting at y_0 (γ' is unique)

$f(\tilde{x}) = \gamma'(1)$ (endpoint of γ').

Step 2 f is well-defined

$\delta^{\hat{x}}$ another path from $\hat{x}_0$ to $\hat{x}$

$\rightsquigarrow \delta^x = p \circ \delta^{\hat{x}}$, δ' lift of δ^x starting at y_0

γ^x, δ^x : paths from x_0 to $p(\hat{x})$

$\rightsquigarrow \gamma^x \cdot \underbrace{\tilde{\delta}^x}_{\text{inv. path}} = p \circ (\gamma^{\hat{x}} \cdot \tilde{\delta}^{\hat{x}})$ loop at x_0

(in the img of p_*)

$\underset{\operatorname{img} p_* \subset \operatorname{img} q_*}{\Rightarrow}$ $\exists$ lift h' of $h \underset{\text{path homotop}}{\simeq} \gamma^x \cdot \tilde{\delta}^x$ as a loop based at y_0

lift of path homotop. shows : the lift of $\gamma^x \cdot \tilde{\delta}^x$ is also a loop at y_0.

uniqueness of lifts : γ' and δ' have the same endpts

Step 3 f is cont.

fix $\hat{x} \in \hat{X}$, neigh. U of $p(\hat{x})$ evenly covered by p and q. $\hat{U} \subset \hat{X}$ comp. of $p^{-1}(U)$ containing $\hat{x}$.

$V \subset Y$ comp of $q^{-1}(U)$ containing $f(\hat{x})$

$\rightsquigarrow f|_{\hat{U}} : \hat{U} \to V$ is homeo (identity $\hat{U}, V$ with U) $\square$

Realization of univ. covering

X locally simply conn. ($\forall x \in X$, $U \underset{\text{open}}{\ni} x$ $\exists x \in V \subset U$ V simpl. conn.)

x_0 fixed basept in X

$\Rightarrow \hat{X} = \{ [\gamma] : \gamma$ path in X, starting at $x_0 \}$

$p : \hat{X} \to X$, $[\gamma] \mapsto \gamma(1)$

$\hat{x}_0 = [$ const path at $x_0]$

2023.4-26 no.3

Topology on $\tilde{X}$

V simply conn. open neigh of x.

γ : path from x_0 to x.

$\rightsquigarrow$ $\tilde{V} = \{[\gamma \cdot \gamma'] : \gamma'$ path $\underset{\text{in } V}{\wedge}$ from x to $x' \in V\}$

$p|_{\tilde{V}} : \tilde{V} \to V$ is bijective.

the sets of the form $\tilde{V}$ form a basis of topology on $\tilde{X}$

Example $X = \mathbb{R}/\mathbb{Z}$. $x_0 = [0]$,

$\tilde{X} = \mathbb{R}$ $\tilde{x}_0 = 0$

paths from $[0]$ to $[t]$ lift to paths from 0 to $t+n$. $n \in \mathbb{Z}$

$Y = \mathbb{R}/k\mathbb{Z}$ $k = 2, 3, 4, \cdots$ $y_0 = [0]_Y$

$q : Y \to X$ $[t]_Y \mapsto [t]_X$.

$q_*(\pi_1(Y, y_0)) \subset \pi_1(X, x_0)$ corr. to $k\mathbb{Z} \subset \mathbb{Z}$.

generator of $\pi_1(Y, y_0)$ is $\gamma(t) = [kt]$ $(0 \le t \le 1)$

Classification of covering spaces

<u>Galois correspondence</u> (Th'm 1.38)

$\{$ subgroups $H < \pi_1(X, x_0) \}$

$\underset{1:1}{\longleftrightarrow}$ $\{$ covering spaces $q : Y \to X$ & $y_0 \in q^{-1}(x_0) \}$

$\{$ subgrps H up to conjugacy $\}$

$\underset{1:1}{\longleftrightarrow}$ $\{$ cov. sp. $q : Y \to X \}$.

Ex. $\boxed{X = \mathbb{R}/\mathbb{Z}}$ subgrps of $\mathbb{Z}$: $k\mathbb{Z}$ $k = 0, 1, 2, \cdots$

$k=0$ corr to $\mathbb{R}$ $k > 0$ corr. to $\mathbb{R}/k\mathbb{Z}$