combinatorial structure behind Δ-cplxes, singular homology

Recall: a Δ-cplx X is given by

- for $n = 0, 1, 2, \cdots$: index set I_n of n-dim simplexes in X.
- $\sigma_\alpha : \Delta^n \to X$ coord map of the α-th simplex $\alpha \in I_n$

s.t. $\forall$ face $\Delta^n_j = [v_0, \cdots \hat{v}_j, \cdots v_n] \subset \Delta^n$, $\alpha \in I_n$

$\exists! \beta \in I_{n-1} : \sigma_\alpha|_{\Delta^n_j} = \sigma_\beta$ up to can. coord map $\Delta^{n-1} \to \Delta^n_j$

This means: for each $j = 0, \cdots, n$ we have

a map $d_{n,j} : I_n \to I_{n-1}$

Consistency condition (two ways to reach $[v_0, \cdots, \hat{v}_i \cdots \hat{v}_j \cdots v_n]$)

$$d_{n-1,i}\, d_{n,j} = d_{n-1,j-1}\, d_{n,i} \quad (i < j)$$

Such structure $(I_n, d_{n,i} : I_n \to I_{n-1})_{n, 0 \le i \le n}$

is called a <u>Semisimplicial set</u>.

Formally: Δ_+: category of
- nonempty finite ordered sets
- injective ord. pres. maps

Semisimplicial set = contrav. functor $\Delta_+ \xrightarrow{F} \mathrm{Set}$

$I_n \longleftrightarrow F([n])$ $\quad [n] = \{0, \cdots, n\}$

$d_{n,i} \longleftrightarrow F(\delta_{n,i})$ $\quad \delta_{n,i} : [n-1] \to [n]$, "avoid i"

Variation: <u>simplicial set</u> $F : \Delta^{op} \to \mathrm{Set}$

Δ: category with same obj's as Δ_+

- ord. pres. maps (not nec. inj.)

ex. $\sigma_{n,i} : [n+1] \to [n]$, "repeat i and $i+1$".

$\rightsquigarrow (I_n, d_{n,i}, s_{n,i} : I_n \to I_{n+1})_{n, 0 \le i \le n}$

- Δ-cplx $X \rightsquigarrow$ simplicial homology $H^\Delta_\bullet(X)$ is defined by $C^\Delta_n(X) = \mathbb{Z} I_n$; lin. comb of I_n
  bdry $d = \sum_{i=0}^{n} (-1)^i d_{n,i} : \mathbb{Z} I_n \to \mathbb{Z} I_{n-1}$

- top sp $X \rightsquigarrow$ singular homology $H_\bullet(X)$ is defined by $C_n(X) = \mathbb{Z}\,\mathrm{Sing}_n(X)$ ....

Prop. $(\mathrm{Sing}_n(X))_n$ is a simplicial set

Proof. Want : $f^* : \mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$ for
$f : [m] \to [n]$ mor. in Δ

- $f_\# : \Delta^m \to \Delta^n$ the unique affine map
  s.t. $f_\#(v_i) = v_{f(i)}$.
  $f_\#\left(\sum_{i=0}^{m} t_i v_i\right) = \sum_i t_i v_{f(i)}$ or
  $f(t_0, \dots, t_m) = (0, \dots, \underset{f(0)}{t_0}, 0, \underset{f(1)}{t_1}, \dots)$
- $f^*(\sigma) = \sigma \circ f_\#$ for $\sigma \in \mathrm{Sing}_n(X)$ $(\Delta^n \xrightarrow{\sigma} X)$
- $[a] \xrightarrow{f} [b] \xrightarrow{g} [c] \rightsquigarrow (gf)_\# = g_\# f_\#$.
  compos of affine maps = affine.

Cor. the singular chain cplx $C_\bullet(X)$ comes from the semisimpl. set $\Delta_+ \underset{\text{ind}}{\to} \Delta \xrightarrow{\mathrm{Sing}_\bullet(X)} \mathrm{Set}$

Geometric realization

Recall : Δ-cplx X can be constructed from the assoc. semisimpl. set $(I_n, d_{n,i})$ by

- $X^0 = I_0$ (vertex set)
- $X^n = \left(X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} \Delta^n\right)\right) /$ glueing from $d_{n,i}\ I_n \to I_{n-1}$.
- $X = \bigcup X^n$.

More formally : $X = \left(\bigcup_n \Delta^n \times I_n\right) / \sim$.

$\sim$: for $f : [m] \to [n]$ in Δ_+, $\underset{\Delta^m \times I_m}{(x, f^*(\alpha))} \sim \underset{\Delta^n \times I_n}{(f_\#(x), \alpha)}$ ($f_\#(x)$ in bdry)

2023-5-10 no. 3

similarly : $F : \Delta^{op} \to Set$ simpl. set

$\rightsquigarrow$ the space $|F| = (\bigcup_n \Delta^n \times I_n) / \sim$

$\sim$ from morphisms in Δ.

is called the (thin) geometric realization of F

Rem $\|F\| = (\bigcup_n \Delta^n \times I_n) / \sim_{\text{from } \Delta_+}$ is called the

fat geom. realization.

$\|F\| \to |F|$ is homotopy equiv. ...

Example. group $G \rightsquigarrow BG = |G^\circ|$.

$(G^n)_{n=0}^{\infty}$ as a simpl. set :

$d_{n,i}(g_1, \cdots g_n) =$ drop g_1 or g_n, or

$(\cdots g_i, g_{i+1}, \cdots) \mapsto (\cdots, g_i g_{i+1}, \cdots)$

$s_{n,i}(g_1, \cdots g_n) = (g_1, \cdots, \underset{i+1}{e}, g_{i+1}, \cdots)$

Rem. $Mor_{Top}(|F|, Y) \cong Mor_{sSet}(F, Sing_\cdot(Y))$.

Simplicial space.

We can consider $X_\cdot : \Delta^{op} \to Space$

(Space = Top, CGHaus, sSet, $\cdots$)

and make sense of

$|X_\cdot| = (\bigcup_{n=0}^{\infty} \Delta^n \times X_n) / \sim_{\text{from } \Delta}$.

$\|X_\cdot\| = \cdots / \sim_{\text{from } \Delta_+}$

(in sSet Δ^n is given by $Mor_\Delta(\sim, [n])$).

Ex. topological group $G \rightsquigarrow (G^n)_{n=0}^{\infty}$ simpl. sp.

$BG = |G^\circ|$ classifying space (for G-bundles)