5th March, 2024

MAT4530

Mandatory assignment 1 of 1

Submission deadline

Thursday 11th April 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

The assignment consists of 8 problems, each worth 10 points. A total score of 40 %, or 32 points, is sufficient to pass.

Let X be a space and $n \geq 0$ an integer. The n-th symmetric product

$$\mathrm{SP}_n(X) = X^n/\Sigma_n$$

is the orbit space for the permutation action

$$\sigma \cdot (x_1, x_2, \dots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)})$$

of the symmetric group Σ_n on the n-fold Cartesian product $X^n = X \times X \times \cdots \times X$. Equivalently, it is the quotient space of X^n by the equivalence relation

$$(x_1, x_2, \dots, x_n) \sim_n (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)})$$

where $(x_1, x_2, \dots, x_n) \in X^n$ and $\sigma \in \Sigma_n$. Take as known that if X is compact Hausdorff, then so is $\mathrm{SP}_n(X)$.

Problem 1. In the case $X = [0, 1] \subset \mathbb{R}$, each $\sim_n$-equivalence class in X^n contains a unique element $(x_1, x_2, \dots, x_n) \in [0, 1]^n$ with

$$1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\,.$$

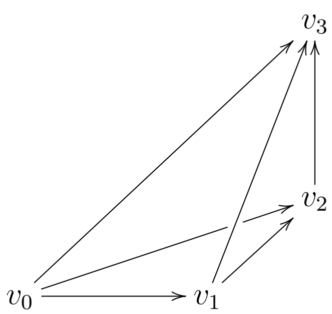
Use this to show that there is a homeomorphism

$$\mathrm{SP}_n([0, 1]) \cong [v_0, v_1, \dots, v_n]\,.$$

Here $[v_0, v_1, \dots, v_n]$ denotes the n-simplex in $\mathbb{R}^n$ spanned by the vertices

$$v_i = (1, \dots, 1, 0, \dots, 0) \in \mathbb{R}^n$$

with i copies of 1 followed by $(n - i)$ copies of 0, for each $0 \leq i \leq n$.



Problem 2. View $S^1 \cong [0,1]/(0 \sim 1)$ as a quotient space of $[0,1]$. Show that there is a homeomorphism $\mathrm{SP}_n(S^1) \cong Y_n$ where

$$Y_n := [v_0, v_1, \ldots, v_n]/(\approx)$$

is the Δ-complex obtained from the n-simplex $[v_0, v_1, \ldots, v_n]$ by identifying the $(b-a)$-face $[v_a, v_{a+1}, \ldots, v_b]$ with the $(b-a)$-face

$$[v_0, v_1, \ldots, v_{b-a}]\,,$$

whenever $0 < a < \cdots < b \leq n$.

This has the consequence that the k-faces $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}]$ and $[v_0, v_{i_1-i_0}, \ldots, v_{i_k-i_0}]$ are identified, whenever $0 \leq i_0 < i_1 < \cdots < i_k \leq n$. Let

$$[w_{i_0}, w_{i_1}, \ldots, w_{i_k}] \in \Delta_k(Y_n)$$

denote the image of $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}] \in \Delta_k([v_0, v_1, \ldots, v_n])$.

Problem 3. Show that $\mathrm{SP}_2(S^1) \cong Y_2$ is a Möbius band.

Problem 4. Determine the groups and homomorphisms in the simplicial chain complex

$$0 \to \Delta_2(Y_2) \xrightarrow{\partial_2} \Delta_1(Y_2) \xrightarrow{\partial_1} \Delta_0(Y_2) \to 0$$

and use this to give presentations (with generators and relations, if any) of the simplicial homology groups $H_m^\Delta(Y_2)$ for $0 \leq m \leq 2$. Consider using the shorthand notation $[i_0 i_1 \cdots i_k]$ for $[w_{i_0}, w_{i_1}, \ldots, w_{i_k}]$.

Problem 5. Determine the groups and homomorphisms in the simplicial chain complex

$$0 \to \Delta_3(Y_3) \xrightarrow{\partial_3} \Delta_2(Y_3) \xrightarrow{\partial_2} \Delta_1(Y_3) \xrightarrow{\partial_1} \Delta_0(Y_3) \to 0$$

and use this to give presentations (with generators and relations, if any) of the simplicial homology groups $H_m^\Delta(Y_3)$ for $0 \leq m \leq 3$.

Problem 6. For $n \geq 2$, let $L_{n-1} = [v_0, \ldots, v_{n-1}] \cup [v_1, \ldots, v_n]$ be the union of the 0-th and the n-th face of $[v_0, \ldots, v_n]$ along their intersection $[v_1, \ldots, v_{n-1}]$. Show that L_{n-1} is a deformation retract of $[v_0, \ldots, v_n] \cong \Delta^n$.

Problem 7. For $n \geq 2$ there is a pushout square

$$\begin{tikzcd} L_{n-1} \arrow[r, tail, "i"] \arrow[d, two heads] & {[v_0, \dots, v_n]} \arrow[d, two heads] \\ Y_{n-1} \arrow[r, tail, "i'"] & Y_n \, , \end{tikzcd}$$

so that $Y_n \cong Y_{n-1} \cup_{L_{n-1}} [v_0, \dots, v_n]$. Deduce from (6) that $i' \colon Y_{n-1} \to Y_n$ is a homotopy equivalence for each $n \geq 2$.

Problem 8. Give presentations of the homology groups $H_m^{\Delta}(Y_n)$ for all $0 \leq m \leq n$ and $n \geq 1$. In these terms, what is the homology class in $H_1^{\Delta}(Y_n)$ of the 1-cycle $[w_0, w_n]$?