SAMPLE SOLUTIONS TO MANDATORY ASSIGNMENT, MAT4530 SPRING 2024

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(1) The n-simplex $[v_0, \dots, v_n] \subset \mathbb{R}^n$ consists of the convex linear combinations $\sum_{i=0}^n t_i v_i$ with $t_i \geq 0$ and $\sum_{i=0}^n t_i = 1$, which equals the subspace of points $(x_1, \dots, x_n)$ with $1 \geq x_1 \geq \cdots \geq x_n \geq 0$. Here $x_j = \sum_{i=j}^n t_i$ for each $1 \leq j \leq n$, so $t_0 = 1 - x_1$, $t_j = x_j - x_{j+1}$ and $t_n = x_n$. The composite

$$[v_0, \dots, v_n] \subset [0,1]^n \longrightarrow \mathrm{SP}_n([0,1])$$

is thus a continuous bijection of compact Hausdorff spaces, and is therefore a homeomorphism.

(2) The surjection $[0,1] \to S^1$ induces the surjection $\mathrm{SP}_n([0,1]) \to \mathrm{SP}_n(S^1)$ that identifies the image of

$$1 = \cdots = 1 > x_{a+1} \geq \cdots \geq x_b > 0 = \cdots = 0$$

in $[v_0, \dots, v_n]$ (with a ones and $n-b$ zeros) with the image of

$$x_{a+1} \geq \cdots \geq x_b > 0 = \cdots = 0 = 0 = \cdots = 0$$

in $[v_0, \dots, v_n]$ (with 0 ones and $n-b+a$ zeros). This means that the face $[v_a, \dots, v_b]$ is identified with $[v_0, \dots, v_{b-a}]$, preserving the barycentric coordinates, so that $\mathrm{SP}_n(S^1)$ is the quotient Δ-complex obtained from $[v_0, \dots, v_n] \cong \Delta^n$ by making these identifications.

Such identifications also imply identifications of the lower-dimensional faces, so that for $a = i_0 < i_1 < \cdots < i_k = b$ the k-face $[v_{i_0}, v_{i_1}, \dots, v_{i_k}]$ is identified with the k-face $[v_0, v_{i_1-i_0}, \dots, v_{i_k-i_0}]$ corresponding to $0 = i_0 - a < i_1 - a < \cdots < i_k - a = b - a$.

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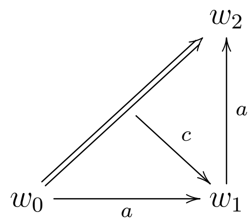
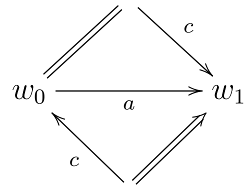
(3) In the triangle $[w_0, w_1, w_2]$, with the edges $[w_0, w_1]$ and $[w_1, w_2]$ identified to a,



we cut along c, flip one triangle over and glue along a, to obtain



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which is a model for the Möbius band.

(4) Let $[012] = [w_0, w_1, w_2]$, etc. The chain complex is

$$0 \to \mathbb{Z}\{[012]\} \xrightarrow{\partial_2} \mathbb{Z}\{[01] = [12], [02]\} \xrightarrow{\partial_1} \mathbb{Z}\{[0] = [1] = [2]\} \to 0$$

with

$$\begin{aligned} \partial_2([012]) &= [12] - [02] + [01] = 2 \cdot [01] - [02] \\ \partial_1([01]) &= [1] - [0] = 0 \\ \partial_1([02]) &= [2] - [0] = 0 \,. \end{aligned}$$

Here $\mathrm{im}(\partial_1) = 0$, $\ker(\partial_1) = \Delta_1(Y^2)$, $\mathrm{im}(\partial_2) = \mathbb{Z}\{2 \cdot [01] - [02]\}$ and $\ker(\partial_2) = 0$. Hence $H_0^\Delta(Y_2) = \Delta_0(Y^2) \cong \mathbb{Z}$ generated by $[0]$, $H_1^\Delta(Y_2) = \Delta_1(Y^2)/\mathbb{Z}\{2 \cdot [01] - [02]\} \cong \mathbb{Z}$ generated by $[01]$, and $H_2^\Delta(Y_2) = 0$.

(5) The chain complex is

$$0 \to \mathbb{Z}\{[0123]\} \xrightarrow{\partial_3} \mathbb{Z}\{[012], [013], [023]\} \xrightarrow{\partial_2} \mathbb{Z}\{[01], [02], [03]\} \xrightarrow{\partial_1} \mathbb{Z}\{[0]\} \to 0$$

with

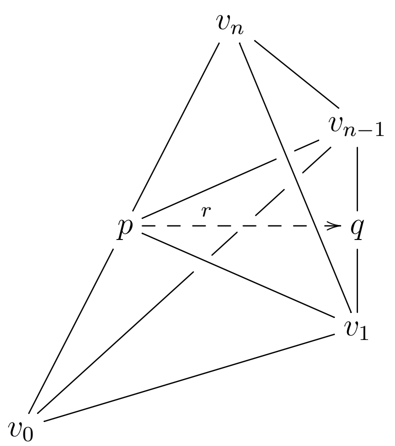
$$\begin{aligned}
\partial_3([0123]) &= [123] - [023] + [013] - [012] = [013] - [023] \\
\partial_2([012]) &= [12] - [02] + [01] = 2 \cdot [01] - [02] \\
\partial_2([013]) &= [13] - [03] + [01] = [01] + [02] - [03] \\
\partial_2([023]) &= [23] - [03] + [02] = [01] + [02] - [03] \\
\partial_1([01]) &= [1] - [0] = 0 \\
\partial_1([02]) &= [2] - [0] = 0 \\
\partial_1([03]) &= [3] - [0] = 0\,.
\end{aligned}$$

Here $H_0^\Delta(Y_3) \cong \mathbb{Z}$ generated by [0], $H_1^\Delta(Y_3) \cong \mathbb{Z}$ generated by [01], $H_2^\Delta(Y_3) = 0$ and $H_3^\Delta(Y_3) = 0$.

(6) Let $p \in [v_0, v_n]$ and $q \in [v_1, \dots, v_{n-1}]$ be interior points, e.g. the barycenters, of the indicated 1-face and $(n-1)$-face, respectively. Then $[v_0, \dots, v_n]$ is the union of the two n-simplices $[v_0, p, v_1, \dots, v_{n-1}]$ and $[p, v_n, v_1, \dots, v_{n-1}]$, which meet along $[p, v_1, \dots, v_{n-1}]$. We define a retraction

$$r\colon [v_0, \dots, v_n] \longrightarrow L_{n-1}$$

to be affine linear on each of these n-simplices, sending p to q in both cases, and taking each vertex v_i to itself (as is required of a retraction to a subspace containing all of these vertices).



Since $[v_0, \dots, v_n]$ is convex, this is a deformation retraction. More explicitly, the convex linear expression

$$H(x,t) = (1-t)x + tir(x)$$

defines a homotopy $H\colon [v_0, \dots, v_n] \times [0,1] \to [v_0, \dots, v_n]$ from the identity to the composite $ir\colon [v_0, \dots, v_n] \to L_{n-1} \to [v_0, \dots, v_n]$.

(7) By passage to quotient spaces along $[v_0, \dots, v_n] \to Y_n$, the retraction r induces a retraction $r' \colon Y_n \to Y_{n-1}$, and the homotopy H induces a homotopy

$$H' \colon Y_n \times [0, 1] \longrightarrow Y_n$$

from the identity to the composite $i'r' \colon Y_n \to Y_{n-1} \to Y_n$. Hence Y_{n-1} is a deformation retract of Y_n, and i' is a homotopy equivalence.

(8) By (7) and homotopy invariance of homology, $i'_* \colon H^\Delta_m(Y_{n-1}) \to H^\Delta_m(Y_n)$ is an isomorphism for each $m \geq 0$ and $n \geq 2$. Hence $\mathbb{Z}\{[0]\} = H^\Delta_0(Y_1) \cong H^\Delta_0(Y_n)$, $\mathbb{Z}\{[01]\} = H^\Delta_1(Y_1) \cong H^\Delta_1(Y_n)$, and $0 = H^\Delta_m(Y_1) \cong H^\Delta_m(Y_n)$ for all $m \geq 2$.

We prove by induction on $n \geq 1$ that the 1-cycle $[w_0, w_n]$ in $\Delta_*(Y_n)$ is homologous to n times the 1-cycle $[w_0, w_1]$ generating $H^\Delta_1(Y_n)$. This is clear for $n = 1$. Suppose it holds for $(n - 1)$, so that $[w_0, w_{n-1}]$ is homologous to $(n - 1)[w_0, w_1]$ in $\Delta_*(Y_{n-1})$. The same relation then holds in $\Delta_*(Y_n)$, since i' induces a chain map. Moreover,

$$\partial_2([w_0, w_{n-1}, w_n]) = [w_{n-1}, w_n] - [w_0, w_n] + [w_0, w_{n-1}]$$

in $\Delta_1(Y_n)$, so $[w_0, w_n]$ is homologous to $[w_0, w_{n-1}] + [w_{n-1}, w_n]$. Here $[w_{n-1}, w_n] = [w_0, w_1]$ in Y_n, so this is homologous to $(n - 1)[w_0, w_1] + [w_0, w_1] = n[w_0, w_1]$. This completes the inductive step.