EXCISION

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Let A and B be subspaces of X, and suppose that their interiors $U = \text{int}(A)$ and $V = \text{int}(B)$ cover X, so that $X = U \cup V = A \cup B$. Say that a singular n-chain $\sum_i n_i \sigma_i$ in X is *fine* (with respect to $\{A, B\}$) if each σ_i has image contained in A or B. Let $C_n(A + B) \subset C_n(X)$ be the subgroup of fine singular n-chains. The boundary of a fine n-chain is a fine $(n-1)$-chain, so $(C_*(A+B), \partial)$ is a subcomplex of $(C_*(X), \partial)$. Let

$$\iota\colon C_*(A + B) \longrightarrow C_*(X)$$

be the inclusion of that subcomplex. Let $H_n(A + B) = H_n(C_*(A + B), \partial)$ be the homology groups of the subcomplex of fine chains.

Theorem 1 (Fine Chains)**.** *The inclusion ι induces an isomorphism*

$$\iota_*\colon H_n(A + B) \xrightarrow{\cong} H_n(X)$$

for each integer n.

Proof. We construct a *subdivision operator* $S\colon C_n(X) \to C_n(X)$ for each n, and show that this is a chain map that is chain homotopic to the identity, by a chain homotopy $T\colon C_n(X) \to C_{n+1}(X)$ with $\partial T + T\partial = 1 - S$. We arrange that S and T restrict to fine operators $S\colon C_n(A + B) \to C_n(A + B)$ and $T\colon C_n(A + B) \to C_{n+1}(A + B)$, respectively. Then we show that for each simplex $\sigma\colon \Delta^n \to X$ there exists an $m \geq 0$ such that $S^m\sigma$ is fine. It follows that for each chain $\alpha \in C_n(X)$ there exists an $m \geq 0$ such that $S^m\alpha \in C_n(A+B)$. Notice that $D = T + TS + \dots TS^{m-1}$ is a chain homotopy from S^m to the identity, and that it restricts to a fine operator $D\colon C_n(A + B) \to C_{n+1}(A + B)$.

Consider any n-cycle $\alpha \in Z_n(X) \subset C_n(X)$, and choose m so that $S^m\alpha$ is fine. Then $S^m\alpha = \alpha - \partial D\alpha$ represents the same homology class as α. Since $S^m\alpha$ is fine, it follows that ι_* maps the homology class of $S^m\alpha \in Z_n(A + B)$ to the homology class of α, so ι_* is surjective.

Consider any fine n-cycle $\alpha \in Z_n(A + B) \subset C_n(A + B)$, and suppose that ι_* maps the homology class of α to zero, i.e., that $\alpha = \partial\beta$ for a $\beta \in C_{n+1}(X)$. Choose m so that $S^m\beta$ is fine. Then $\partial S^m\beta = S^m\partial\beta = S^m\alpha = \alpha - \partial D\alpha$, where $D\alpha$ is fine. Hence $\alpha = \partial(S^m\beta + D\alpha)$ lies in $B_n(A + B)$ and represents zero in $H_n(A + B)$. Thus ι_* is injective.

We shall initially define S and T on the standard simplices $\Delta^n = [e_0, \dots, e_n]$ for $n \geq 0$, and thereafter extend to general singular simplices $\sigma\colon \Delta^n \to X$ in a "natural" manner. The definitions will be inductively given in the wider generality of *linear simplices* in $\mathbb{R}^\infty$, i.e., singular simplices $\sigma\colon \Delta^n \to \mathbb{R}^\infty$ given by the order-preserving affine linear maps taking $e_0, \dots, e_n$ to given points $v_0, \dots, v_n$. We shall write $[v_0, \dots, v_n]$ for this linear simplex, also in the cases where $v_0, \dots, v_n$ are not in general position. A finite sum of linear simplices, with integer coefficients, will be called a *linear chain*.

For any linear n-simplex $\sigma = [v_0, \dots, v_n]$ and any point b let the *join* of b and σ be the linear $(n + 1)$-simplex

$$b\sigma = [b, v_0, \dots, v_n]\,.$$

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Extend the rule $\sigma \mapsto b\sigma$ to linear chains $\lambda = \sum_i n_i \sigma_i$ by additivity, so that $b\lambda = \sum_i n_i(b\sigma_i)$. Then

$$\partial(b\sigma) = \partial[b, v_0, \dots, v_n] = [v_0, \dots, v_n] - \sum_{i=0}^{n} (-1)^i [b, v_0, \dots, \widehat{v_i}, \dots, v_n] = \sigma - b\partial\sigma\,,$$

where $b\partial\sigma$ is interpreted as $[b]$ for $n = 0$. Hence $\partial b + b\partial = 1 - [b]\epsilon$.

Given any linear n-simplex $\sigma = [v_0, \dots, v_n]$, let

$$b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

be its *barycenter*. It is the point whose $n+1$ barycentric coordinates $(t_0, \dots, t_n) = (1/(n+1), \dots, 1/(n+1))$ are all equal (and sum to 1).

We now define the *subdivision* operator S on linear chains. Each linear 0-chain is its own subdivision: we define $S(\sigma) = \sigma$ for $\sigma = [v_0]$, and extend additively to linear 0-chains. For $n \geq 1$, assume that the subdivision $S(\lambda)$ has been defined for all linear $(n-1)$-chains, including $\lambda = \partial\sigma$. Then for each linear n-simplex σ we let

$$S(\sigma) = b_\sigma S(\partial\sigma)\,.$$

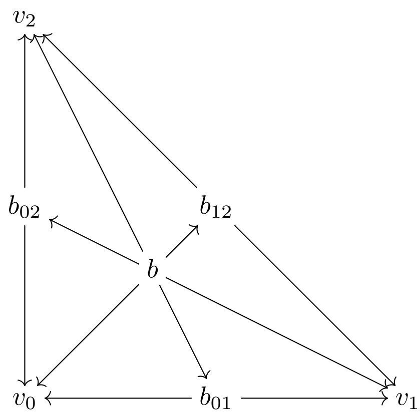
As usual, we extend S additively to linear n-chains. For example,

$$S([v_0, v_1]) = b([v_1] - [v_0]) = [b, v_1] - [b, v_0]$$

where b is the barycenter of $[v_0, v_1]$. Continuing,

$$\begin{aligned} S([v_0, v_1, v_2]) &= bS([v_1, v_2]) - bS([v_0, v_2]) + bS([v_0, v_1]) \\ &= [b, b_{12}, v_2] - [b, b_{12}, v_1] - [b, b_{02}, v_2] \\ &\quad + [b, b_{02}, v_0] + [b, b_{01}, v_1] - [b, b_{12}, v_0]\,, \end{aligned}$$

where b is the barycenter of $[v_0, v_1, v_2]$, and b_{ij} is the barycenter of $[v_i, v_j]$.



The subdivision operator commutes with the boundary operators, i.e., $\partial S(\lambda) = S\partial(\lambda)$. This is clear on linear 0-chains, and to prove it for a linear n-simplex σ we may assume that it holds for all linear $(n-1)$-chains, including $\partial\sigma$. Then

$$\partial S(\sigma) = \partial b_\sigma S(\partial\sigma) = S(\partial\sigma) - b_\sigma \partial S(\partial\sigma) = S(\partial\sigma) - b_\sigma S(\partial\partial\sigma) = S(\partial\sigma)\,.$$

Notice that for each linear n-simplex $\sigma = [v_0, \dots, v_n]$, the subdivision $S(\sigma)$ is a signed sum of linear n-simplices τ, each with image contained in (the image of) σ. For later use, we note that the diameter of (the image of) each τ, with respect to the Euclidean metric in $\mathbb{R}^\infty$, is at most $n/(n+1)$ times that of σ:

$$\mathrm{diam}(\tau) \leq \frac{n}{n+1}\,\mathrm{diam}(\sigma)\,.$$

To see this, note first that the diameter of τ is the distance between two of its vertices. If both of these lie in a proper face of σ, we are done by induction on n, since $n/(n+1)$ increases with n. Otherwise, one of the two vertices is the barycenter b of σ, and we may assume that the other vertex is one of the vertices v_i of σ. Now b lies $n/(n+1)$-th of the way from v_i to the barycenter of the opposite face, so the distance from v_i to b is bounded by $n/(n+1)$ times the diameter of σ, as claimed.

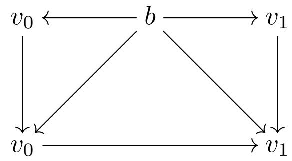
We continue by defining the chain homotopy T on linear chains. For $n = 0$ we let $T(\sigma) = [v_0, v_0]$ for $\sigma = [v_0]$, and extend additively to all linear 0-chains. For $n \geq 1$ assume that $T(\lambda)$ has been defined for all linear $(n-1)$-chains, including $\lambda = \partial\sigma$. Then for each linear n-simplex σ we let

$$T(\sigma) = b_\sigma(\sigma - T(\partial\sigma))\,.$$

Again, we extend T additively to linear n-chains. For example,

$$T([v_0, v_1]) = b([v_0, v_1] - T([v_1] - [v_0])) = [b, v_0, v_1] - [b, v_1, v_1] + [b, v_0, v_0]$$

where b is the barycenter of $[v_0, v_1]$.



We prove that $\partial T + T\partial = 1 - S$ on linear n-chains by induction on n. For $n = 0$, this is the true assertion $\partial[v_0, v_0] = [v_0] - [v_0]$. Let $n \geq 1$ and assume that $\partial T + T\partial = 1 - S$ on linear $(n-1)$-chains. In particular, for any linear n-simplex σ, we know that $\partial T(\partial\sigma) + T(\partial\partial\sigma) = \partial\sigma - S(\partial\sigma)$, so $\partial(\sigma - T(\partial\sigma)) = S(\partial\sigma)$. Then

$$\begin{aligned}\partial T(\sigma) &= \partial b_\sigma(\sigma - T(\partial\sigma)) = (\sigma - T(\partial\sigma)) - b_\sigma\partial(\sigma - T(\partial\sigma))\\ &= \sigma - T(\partial\sigma) - b_\sigma S(\partial\sigma) = \sigma - T(\partial\sigma) - S(\sigma)\,.\end{aligned}$$

Hence $\partial T + T\partial = 1 - S$ on σ, and therefore also on general linear n-chains.

Now we extend the operators S and T to singular chains in X. For $\sigma\colon \Delta^n \to X$ define $S(\sigma) \in C_n(X)$ by

$$S(\sigma) = \sigma_\# S(\Delta^n)\,.$$

Here $S(\Delta^n)$ is a signed sum of linear n-simplices $\Delta^n \to \Delta^n$ in $\Delta^n \subset \mathbb{R}^\infty$; by $\sigma_\# S(\Delta^n)$ we mean the corresponding signed sum of singular simplices in X given by composing σ with these linear simplices. For example, when $n = 1$,

$$S(\sigma) = \sigma|[b, v_1] - \sigma|[b, v_0]$$

where b is the barycenter of $[v_0, v_1]$, and each restriction is implicitly composed with the order-preserving affine linear homeomorphism $[e_0, e_1] \to [b, v_1]$, or $[e_0, e_1] \to [b, v_0]$, according to the case. As usual, S is defined on singular n-chains by additivity. It follows from the fact that $\sigma_\#$ is a chain map, $\partial S = S\partial$ on linear chains, and the definitions given, that

$$\partial S(\sigma) = \partial\sigma_\# S(\Delta^n) = \sigma_\# S(\partial\Delta^n) = S(\partial\sigma)\,.$$

Finally, we define $T\colon C_n(X) \to C_{n+1}(X)$ by

$$T(\sigma) = \sigma_\# T(\Delta^n)\,.$$

Here $T(\Delta^n)$ is a signed sum of linear n-simplices in $\Delta^n \subset \mathbb{R}^\infty$, and $\sigma_\# T(\Delta^n)$ denotes the corresponding signed sum of singular simplices in X obtained by composition with $\sigma\colon \Delta^n \to X$. As for S we find that $\sigma_\# T(\partial\Delta^n) = T(\partial\sigma)$, so

$$\partial T(\sigma) = \sigma - \sigma_\# T(\partial\Delta^n) - \sigma_\# S(\Delta^n) = \sigma - T(\partial\sigma) - S(\sigma)\,,$$

and $\partial T + T\partial = 1 - S$ on σ. Hence this identity also holds on general singular n-chains α.

It is clear that if σ has image in A (resp. B), then $S(\sigma)$ and $T(\sigma)$ are signed sums of singular simplices with images in A (resp. B), so if α is fine with respect to $\{A, B\}$, then so are $S(\alpha)$ and $T(\alpha)$.

It remains to show that for each $\sigma\colon \Delta^n \to X$ we can find an $m \geq 0$ such that $S^m\sigma$ is fine. For this, we use the Lebesgue number lemma for the compact space Δ^n, with the Euclidean metric from $\mathbb{R}^{n+1}$, and the open cover $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$. The lemma asserts that there exists an $\epsilon > 0$ such that every subset $Q \subset \Delta^n$ of diameter less than ϵ is contained in $\sigma^{-1}(U) \subset \sigma^{-1}(A)$ or in $\sigma^{-1}(V) \subset \sigma^{-1}(B)$. Equivalently, $\sigma(Q)$ is contained in $U \subset A$ or in $V \subset B$. Hence if Q is a linear simplex within Δ^n, then σ restricted to Q is fine with respect to $\{A, B\}$.

Recall that $S(\Delta^n)$ is a signed sum of linear simplices with images of diameter at most $n/(n+1)$ times $\mathrm{diam}(\Delta^n) = \sqrt{2}$. More generally, $S^m(\Delta^n)$ is a signed sum of linear simplices τ with images of diameter at most $(n/(n+1))^m \cdot \sqrt{2}$. These bounds tend to 0 as m increases to ∞, so there exists an $m \geq 0$ with $(n/(n+1))^m \cdot \sqrt{2} < \epsilon$, where ϵ is a Lebesgue number of $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$. Hence for this m, the subdivision $S^m(\sigma)$ is fine with respect to $\{A, B\}$, as claimed. □

Consider the following inclusion maps.

$$\begin{array}{ccc} A \cap B & \xrightarrow{i_A} & A \\ {\scriptstyle i_B}\big\downarrow & & \big\downarrow{\scriptstyle j_A} \\ B & \xrightarrow{j_B} & X \end{array}$$

Theorem 2 (Mayer–Vietoris)**.** *Let $A, B \subset X$ be subspaces whose interiors cover X. There is a natural long exact sequence*

$$\dots \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \xrightarrow{\Phi} \dots$$

where $\Phi = (i_{A*}, i_{B*})$ and $\Psi = j_{A*} - j_{B*}$.

Proof. In each degree n, the subgroups $C_n(A)$ and $C_n(B)$ of $C_n(X)$ intersect in $C_n(A \cap B)$ and span $C_n(A + B)$. Hence there is a short exact sequence of chain complexes

$$0 \to C_*(A \cap B) \xrightarrow{\phi} C_*(A) \oplus C_*(B) \xrightarrow{\psi} C_*(A + B) \to 0 \tag{1}$$

where $\phi = (i_{A\#}, i_{B\#})$ and $\iota\psi = j_{A\#} - j_{B\#}$. Hence the upper row in the following diagram is exact, and splices together to a long exact sequence as n varies.

$$\begin{array}{ccccccc} H_n(A \cap B) & \xrightarrow{\Phi} & H_n(A) \oplus H_n(B) & \longrightarrow & H_n(A + B) & \longrightarrow & H_{n-1}(A \cap B) \\ & & & {\scriptstyle\Psi}\searrow & \big\downarrow{\scriptstyle \iota_*} & \nearrow{\scriptstyle\partial} & \\ & & & & H_n(X) & & \end{array}$$

By the previous theorem, ι_* is an isomorphism under the topological hypothesis on A and B. Hence the lower row is also exact, and splices together for varying n to a long exact sequence. □

Note that the homomorphism ∂ in the Mayer–Vietoris long exact sequence is given by first inverting ι_*, and then applying the connecting homomorphism for the short exact sequence of chain complexes (1). More explicitly, for an n-cycle $\gamma \in Z_n(X)$ we apply subdivision enough times to ensure that we can write $S^m(\gamma) = \alpha - \beta$ with $\alpha \in C_n(A)$ and $\beta \in C_n(B)$. Then $\partial\alpha = \partial\beta$ is a cycle in $C_{n-1}(A \cap B)$, and $\partial[\gamma] = [\partial\alpha]$ is its homology class.

Theorem 3 (Excision)**.** *Let $A, B \subset X$ be subspaces whose interiors cover X. Then the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms*

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

for all n. Equivalently, if $Z \subset A \subset X$ are such that $\mathrm{cl}(Z) \subset \mathrm{int}(A)$, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)$$

for all n.

Proof. In each degree n, the subgroups $C_n(A)$ and $C_n(B)$ of $C_n(X)$ intersect in $C_n(A \cap B)$ and span $C_n(A + B)$. Hence the inclusion $C_*(B) \to C_*(A + B)$ induces an isomorphism of chain complexes

$$C_*(B)/C_*(A \cap B) \xrightarrow{\cong} C_*(A + B)/C_*(A)\,.$$

We write $C_*(B, A \cap B)$ for the left hand quotient, as usual, and write $C_*(A + B, A)$ for the right hand quotient. With this notation, we have the following vertical maps of horizontal short exact sequences of chain complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(A \cap B) & \longrightarrow & C_*(B) & \longrightarrow & C_*(B, A \cap B) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \cong \big\downarrow & & \\
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(A + B) & \longrightarrow & C_*(A + B, A) & \longrightarrow & 0 \\
 & & = \big\downarrow & & \iota \big\downarrow & & \bar\iota \big\downarrow & & \\
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, A) & \longrightarrow & 0
\end{array}$$

The homomorphism of relative homology groups induced by the inclusion $(B, A \cap B) \to (X, A)$ is thus the composite of the isomorphism

$$H_*(B, A \cap B) \xrightarrow{\cong} H_*(A + B, A)$$

induced by the chain level isomorphism above, and the homomorphism

$$\bar\iota_* \colon H_*(A + B, A) \longrightarrow H_*(X, A)$$

induced by the chain map $\bar\iota \colon C_*(A + B, A) \to C_*(X, A)$. The identity map, ι and $\bar\iota$ induce a vertical map of horizontal long exact sequences

$$\begin{array}{ccccccccc}
H_n(A) & \longrightarrow & H_n(A + B) & \longrightarrow & H_n(A + B, A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & H_{n-1}(A + B) \\
= \big\downarrow & & \iota_* \big\downarrow & & \bar\iota_* \big\downarrow & & = \big\downarrow & & \iota_* \big\downarrow \\
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & H_{n-1}(X)\,.
\end{array}$$

The maps in the first, second, fourth and fifth columns are isomorphisms, by the proposition above in the case of ι_*. Hence, by the five-lemma it follows that the map in the third column, $\bar\iota_*$, is also an isomorphism. Thus $H_*(B, A \cap B) \to H_*(X, A)$ is a composite of two isomorphisms, and is therefore an isomorphism.

The alternative formulation arises by setting $B = X \setminus Z$, since then $\mathrm{int}(B) = X \setminus \mathrm{cl}(Z)$, and $\mathrm{int}(A) \cup \mathrm{int}(B) = X$ is equivalent to $\mathrm{cl}(Z) \subset \mathrm{int}(A)$. □

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