On Lemma 2.18 in Hatcher's "Vector Bundles and K-Theory".

Let $D = D^{2n}$ with boundary $S = S^{2n-1} = \partial D$. Consider S as the equator in

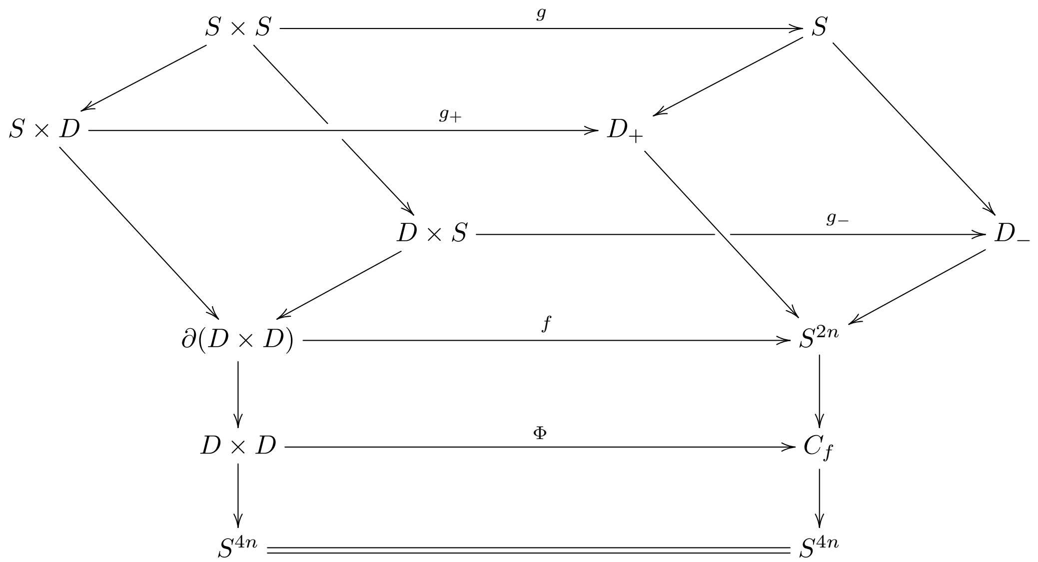
$$S^{2n} = D_+ \cup_S D_- \,,$$

and consider

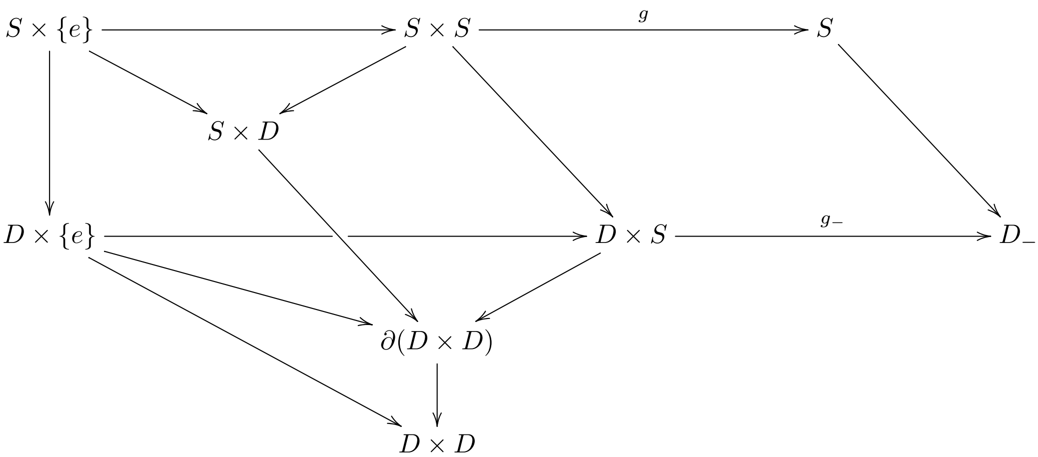
$$\partial(D \times D) = S \times D \cup_{S\times S} D \times S$$

as a model for S^{4n-1}.

Given any map $g\colon S \times S \to S$, the Hopf construction $f = H(g)\colon S^{4n-1} \to S^{2n}$ is defined as the union of any choice of maps $g_+\colon S \times D \to D_+$ and $g_-\colon D \times S \to D_-$ extending g. Furthermore, let $C_f = S^{2n} \cup_f e^{4n}$ be the mapping cone of f, with characteristic map $\Phi\colon D \times D \to C_f$.



Let $e \in S$ be the base point. The inclusion induces maps $S \times \{e\} \to S \times S$ and $D \times \{e\} \to D \times S$ making the following diagram commute.



The same inclusion induces maps $\{e\} \times S \to S \times S$ and $\{e\} \times D \to S \times D$ making another, similar, diagram commute.

We now suppose that the pairing g is left and right unital, with unit element e. Then the composite map $S \times \{e\} \to S \times S \to S$ is the identity, so the map of pairs $(D \times \{e\}, S \times \{e\}) \to (D_-, S)$ is a homotopy equivalence. Likewise, the composite map $\{e\} \times S \to S \times S \to S$ is the identity, so the map of pairs $(\{e\} \times D, \{e\} \times S) \to (D_+, S)$ is also a homotopy equivalence.

The Hopf invariant is defined in terms of the multiplicative properties of the extension

$$0 \to \tilde{K}(S^{4n}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0$$

which we can rewrite as

$$0 \to K(C_f, S^{2n}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0\,.$$

Let $x^n \in \tilde{K}(S^{2n}) \cong \mathbb{Z}$ denote a generator. It can be realized by a generator of $K(S^{2n}, D_+)$, and restricts to a generator of $K(D_-, S)$.

Let $\beta \in \tilde{K}(C_f)$ be an extension of x^n, so that $\beta|S^{2n} = x^n$. It can be realized by a generator of $K(C_f, D_+)$, and pulls back along $\Phi_+ = (\Phi, g_+)\colon (D \times D, S \times D) \to (C_f, D_+)$ to a class $\Phi_+^*(\beta)$ in $K(D \times D, S \times D)$. We claim that the latter class is a generator.

Since D is contractible, the inclusion $(D \times \{e\}, S \times \{e\}) \to (D \times D, S \times D)$ is an equivalence, so it suffices to prove that $\Phi_+^*(\beta)$ restricts to a generator of $K(D \times \{e\}, S \times \{e\})$. Chasing the diagrams, this restriction of $\Phi_+^*(\beta)$ equals the pullback of the generator of $K(D_-, S)$ along the composite map

$$(D \times \{e\}, S \times \{e\}) \longrightarrow (D \times S, S \times S) \xrightarrow{(g_-, g)} (D_-, S)\,.$$

By the right unitality of g, this composite is an equivalence, and hence takes the K-theory generator $x^n|(D_-, S)$ to a K-theory generator.

In the same way, using the left unitality of g, the pullback $\Phi_-^*(\beta)$ along $\Phi_- = (\Phi, g_-) : (D \times D, D \times S) \to (C_f, D_-)$ is a generator of $K(D \times D, D \times S)$.

We now show that the cup product $\beta \cup \beta \in \tilde{K}(C_f)$ is $\pm\alpha$, where α is the image of a generator of $\tilde{K}(S^{4n}) \cong K(C_f, S^{2n}) \cong \mathbb{Z}$. In other words, we need to show that β^2 lifts to a generator of the latter group.

Consider the following commutative diagram, where the relative cup product in the middle row takes values in the topological K-theory of $(C_f, D_+ \cup D_-) = (C_f, S^{2n})$. The relative cup product in the lower row takes values in the topological K-theory of $(D \times D, (S \times D) \cup (D \times S)) = (D \times D, \partial(D \times D))$.

$$\begin{array}{ccc}
\tilde{K}(C_f) \otimes \tilde{K}(C_f) & \xrightarrow{\ \cup\ } & \tilde{K}(C_f) \\
\cong \uparrow & & \uparrow \\
K(C_f, D_+) \otimes K(C_f, D_-) & \xrightarrow{\ \cup\ } & K(C_f, S^{2n}) \\
\downarrow \Phi_+^* \otimes \Phi_-^* & & \cong \downarrow \Phi^* \\
K(D \times D, S \times D) \otimes K(D \times D, D \times S) & \xrightarrow{\ \cup\ } & K(D \times D, \partial(D \times D))
\end{array}$$

Commutativity of the upper square shows that $\beta \cup \beta$ lifts to $K(C_f, S^{2n})$. To prove that it is a generator we apply Φ^*, which is an isomorphism by excision. By the left and right unitality of g the images $\Phi_+^*(\beta)$ and $\Phi_-^*(\beta)$ generate $K(D \times D, S \times D) \cong \mathbb{Z}$ and $K(D \times D, D \times S) \cong \mathbb{Z}$, respectively, so $\Phi_+^*(\beta) \otimes \Phi_-^*(\beta)$ generates the lower left hand group.

Using the cross product

$$K(D, S) \otimes K(D) \xrightarrow{\times} K(D \times D, S \times D)$$

we can write $\Phi_+^*(\beta)$ as $\beta \times 1$, or equivalently as $pr_1^*(\beta)$, where $pr_1\colon (D \times D, S \times D) \to (D, S)$ is given by projection to the first coordinate. Likewise we can write $\Phi_-^*(\beta)$ as $1 \times \beta$, or as $pr_2^*(\beta)$.

Using the commuting triangle

$$\begin{array}{ccc}
K(D \times D, S \times D) \otimes K(D \times D, D \times S) & \xrightarrow{\ \cup\ } & K(D \times D, \partial(D \times D)) \\
\cong \uparrow pr_1^* \otimes pr_2^* & \nearrow_{\cong}^{\times} & \\
K(D, S) \otimes K(D, S) & &
\end{array}$$

we compute that

$$\Phi_+^*(\beta) \cup \Phi_-^*(\beta) = pr_1^*(\beta) \cup pr_2^*(\beta) = \beta \times \beta$$

in $K(D \times D, \partial(D \times D))$. The diagonal arrow is an isomorphism by the Künneth theorem, so $\beta \times \beta$ is indeed a generator of $K(D \times D, \partial(D \times D))$, as we wanted to prove.

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