COROLLARY 6.5. *The infinite projective space* $P^\infty = G_1(R^\infty)$ *is a CW-complex having one* r-*cell* $e(r+1)$ *for each integer* $r \geq 0$. *The closure* $\bar{e}(r+1) \subset P^\infty$ *is equal to the finite projective space* P^r.

The proof is straightforward.

Now let us count the number of r-cells in $G_n(R^m)$ for arbitrary n. It is convenient to introduce the language of partitions.

DEFINITION 6.6. A *partition* of an integer $r \geq 0$ is an unordered sequence $i_1 i_2 \cdots i_s$ of positive integers with sum r. The number of partitions of r is customarily denoted by p(r). Thus for $r \leq 10$ one has the following table.

r	0	1	2	3	4	5	6	7	8	9	10
p(r)	1	1	2	3	5	7	11	15	22	30	42

For example the integer 4 has five partitions, namely: 1 1 1 1, 1 1 2, 2 2, 1 3, and 4. The integer 0 has just one (vacuous) partition. (According to Hardy and Ramanujan the function p(r) is asymptotic to $\exp(\pi\sqrt{2r/3})/4r\sqrt{3}$ as $r \to \infty$. For further information see [Ostmann].)

To every Schubert symbol $(\sigma_1, \ldots, \sigma_n)$ with $d(\sigma) = r$ and $\sigma_n \leq m$ there corresponds a partition $i_1 \cdots i_s$ of r, where $i_1, \ldots, i_s$ denotes the sequence obtained from $\sigma_1 - 1, \ldots, \sigma_n - n$ by cancelling any zeros which may appear at the beginning of this sequence. Clearly

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq m - n$$

and $s \leq n$. Thus

COROLLARY 6.7. *The number of* r-*cells in* $G_n(R^m)$ *is equal to the number of partitions of* r *into at most* n *integers each of which is* $\leq m - n$.

In particular, if both n and m − n are $\geq r$, then the number of r-cells in $G_n(R^m)$ is equal to p(r).

Note that this corollary remains true if m is allowed to take the value $+\infty$.

Here are five problems for the reader.

Problem 6-A. Show that a CW-complex is finite if and only if its underlying space is compact.

Problem 6-B. Show that the restriction homomorphism

$$i^* : H^p(G_n(R^\infty)) \to H^p(G_n(R^{n+k}))$$

is an isomorphism for $p < k$. Any coefficient group may be used. (Compare the description of cohomology for CW-complexes in Appendix A.)

Problem 6-C. Show that the correspondence $X \overset{f}{\mapsto} R^1 \oplus X$ defines an embedding of the Grassmann manifold $G_n(R^m)$ into $G_{n+1}(R^1 \oplus R^m) = G_{n+1}(R^{m+1})$, and that f is covered by a bundle map

$$\varepsilon^1 \oplus \gamma^n(R^m) \to \gamma^{n+1}(R^{m+1}) .$$

Show that f carries the r-cell of $G_n(R^m)$ which corresponds to a given partition $i_1 \cdots i_s$ of r onto the r-cell of $G_{n+1}(R^{m+1})$ which corresponds to the same partition $i_1 \cdots i_s$.

Problem 6-D. Show that the number of distinct Stiefel-Whitney numbers $w_1^{r_1} \cdots w_n^{r_n}[M]$ for an n-dimensional manifold is equal to p(n).

Problem 6-E. Show that the number of r-cells in $G_n(R^{n+k})$ is equal to the number of r-cells in $G_k(R^{n+k})$ [or show that these two CW-complexes are actually isomorphic].