Uniqueness of Stiefel-Whitney Classes

At this point we have not yet shown that there exist Stiefel-Whitney classes $w_i(\xi)$ satisfying the four axioms of §4. Before proving existence, we will prove the following.

UNIQUENESS THEOREM 7.3. *There exists at most one correspondence* $\xi \mapsto w(\xi)$ *which assigns to each vector bundle over a paracompact base space a sequence of cohomology classes satisfying the four axioms for Stiefel-Whitney classes.*

Proof. Suppose that there were two such, say $\xi \mapsto w(\xi)$ and $\xi \mapsto \tilde{w}(\xi)$. For the canonical line bundle γ_1^1 over P^1 we have

$$w(\gamma_1^1) = \tilde{w}(\gamma_1^1) = 1 + a$$

by Axioms 1 and 4. Embedding γ_1^1 in the line bundle γ^1 over the infinite projective space P^∞, it follows that

$$w(\gamma^1) = \tilde{w}(\gamma^1) = 1 + a$$

by Axioms 1 and 2. Passing to the n-fold cartesian product

$$\xi = \gamma^1 \times \ldots \times \gamma^1 \cong \pi_1^*\gamma^1 \oplus \ldots \oplus \pi_n^*\gamma^1 ,$$

it follows that

$$w(\xi) = \tilde{w}(\xi) = (1+a_1)\ldots(1+a_n)$$

by Axioms 2 and 3. Now using the existence of a bundle map $\xi \to \gamma^n$, and the fact that $H^*(G_n)$ injects monomorphically into $H^*(P^\infty \times \ldots \times P^\infty)$, it follows that $w(\gamma^n) = \tilde{w}(\gamma^n)$.

For any n-plane bundle η over a paracompact base space, choosing a bundle map $f : \eta \to \gamma^n$, it follows immediately that

$$w(\eta) = \bar{f}^* w(\gamma^n) = \bar{f}^* \tilde{w}(\gamma^n) = \tilde{w}(\eta) . \blacksquare$$

REMARK. Using essentially this same argument, it would not be difficult to prove a corresponding uniqueness theorem for Stiefel-Whitney classes, working in the much smaller category consisting of smooth vector bundles and smooth bundle mappings, all of the base spaces being smooth paracompact manifolds. It would be much more difficult, however, to prove such a result using only tangent bundles of manifolds. Compare [Blanton and Schweitzer].

Here are three problems for the reader. The first two are based on Problem 6-C.

Problem 7-A. Identify explicitly the cocycle in $C^r(G_n) \cong H^r(G_n)$ which corresponds to the Stiefel-Whitney class $w_r(\gamma^n)$.

Problem 7-B. Show that the cohomology algebra $H^*(G_n(R^{n+k}))$ over $Z/2$ is generated by the Stiefel-Whitney classes $w_1, \ldots, w_n$ of γ^n and the dual classes $\bar{w}_1, \ldots, \bar{w}_k$, subject only to the $n+k$ defining relations

$$(1 + w_1 + \ldots + w_n)(1 + \bar{w}_1 + \ldots + \bar{w}_k) = 1 .$$

(Reference: [Borel, 1953, p. 190].)

Problem 7-C. Let ξ^m and η^n be vector bundles over a paracompact base space. Show that the Stiefel-Whitney classes of the tensor product $\xi^m \otimes \eta^n$ (or of the isomorphic bundle $\mathrm{Hom}(\xi^m, \eta^n)$) can be computed as follows. If the fiber dimensions m and n are both 1, then

$$w_1(\xi^1 \otimes \eta^1) = w_1(\xi^1) + w_1(\eta^1) .$$

More generally there is a universal formula of the form

$$w(\xi^m \otimes \eta^n) = p_{m,n}(w_1(\xi^m), \ldots, w_m(\xi^m), w_1(\eta^n), \ldots, w_n(\eta^n))$$

where the polynomial $p_{m,n}$ in $m+n$ variables can be characterized as follows. If $\sigma_1, \ldots, \sigma_m$ are the elementary symmetric functions of indeterminates $t_1, \ldots, t_m$, and if $\sigma'_1, \ldots, \sigma'_n$ are the elementary symmetric functions of $t'_1, \ldots, t'_n$, then