In MAT4530 (Algebraic Topology I) we studied Chapters 1 and 2 from Hatcher's textbook, on the fundamental group and homology. In MAT4540 (Algebraic Topology II) we will concentrate on Chapters 3 and 4 from Hatcher's textbook, on cohomology and homotopy theory.

Ch. 3: Cohomology

In Algebraic Topology I we studied sums $X \coprod Y$ of spaces. Now we also want to understand products $X \times Y$ of spaces. There is a cross product pairing in homology $H_i(X) \times H_j(Y) \to H_{i+j}(X \times Y)$. For $X = Y$ we also have the diagonal map $\Delta \colon X \to X \times X$, and the covariantly induced homomorphism $\Delta_* \colon H_k(X) \to H_k(X \times X)$. These do not readily compose.

The description of homology groups $H_k(X)$ in terms of chains $\alpha = \sum_i n_i \sigma_i$ goes back to Poincaré in the 1890s. Around 1930, Alexander dualized the construction, replacing chains by cochains $\varphi = \{\sigma \mapsto \varphi(\sigma)\}$, to define cohomology groups $H^k(X)$. These admit a cross product $H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y)$, but are contravariantly functorial, so that Δ induces a homomorphism $\Delta^* \colon H^k(X \times X) \to H^k(X)$. Composing these leads to the cup product $H^i(X) \times H^j(X) \to H^{i+j}(X)$, mapping φ and ψ to $\varphi \cup \psi$. This pairing makes $H^*(X)$ a graded commutative ring.

The universal coefficient theorem expresses $H^*(X)$ in terms of $H_*(X)$, using the functor Ext from homological algebra.

The Künneth theorem expresses $H^*(X \times Y)$ in terms of $H^*(X)$ and $H^*(Y)$, in many cases.

The Poincaré duality theorem concerns the case when $X = M$ is a closed oriented n-dimensional manifold. In this case $H^*(M)$ is self-dual: the cup product $H^k(M) \to H^{n-k}(M) \to H^n(M)$ is a perfect pairing, and there are isomorphisms $H^k(M) \cong H_{n-k}(M)$.

Ch. 4: Homotopy Theory

CW-complexes are inductively built by attaching cells, with n-skeleton $X^{(n+1)} = X^{(n)} \cup_\phi \coprod_i D_i^{n+1}$. The homotopy type only depends on the homotopy class of the attaching map $\phi \colon \coprod_i S^n \to X^{(n)}$, leading to the study of the homotopy classes of maps $S^k \to X$. These define the higher homotopy groups $\pi_k(X)$.

The Hurewicz theorem compares homotopy to homology, showing for $n \geq 2$ that $\pi_n(X) \cong H_n(X)$ if $\pi_k(X) = 0$ for each $k < n$.

The Freudenthal suspension theorem compares $\pi_k(X)$ in a wider range to another generalized homology theory $\pi_k^S(X)$, called stable homotopy theory.

The Eilenberg–Mac Lane representability theorem shows that, for each integer $n \geq 0$ there is a space K_n and a natural isomorphism $H^n(X) \cong [X, K_n] = \{X \to K_n\}/\simeq$. These spaces K_n combine to a spectrum, in the sense of algebraic topology, which is a representing object for ordinary cohomology.

In the spring of 2017, MAT9580 (Algebraic Topology III) may concentrate on stable homotopy theory and the category of spectra, representing generalized cohomology theories. This is the context of "brave new rings" or "structured ring spectra", with the Eilenberg–Mac Lane spectrum corresponding to the classical ring of integers.