MAT4540/9540: EXERCISES 4

In these exercises we will finish the proof that the Grassmannian classifies vector bundles on a compact Hausdorff space X. Let $[X, \mathrm{Gr}_n(\mathbf{F}^N)]$ denote the set of homotopy classes of maps from X to $\mathrm{Gr}_n(\mathbf{F}^N)$, and write $\mathrm{Gr}_n := \mathrm{Gr}_n(\mathbf{F}^\infty) := \bigcup_N \mathrm{Gr}_n(\mathbf{F}^N)$ for the infinite Grassmannian. Furthermore, we let γ^n denote the tautological bundle on Gr_n. Then the classification result reads as follows:

Theorem 1. *Let X be a compact Hausdorff space. With notations as above, there is a bijection*

$$\theta\colon [X, \mathrm{Gr}_n] \to \mathrm{Vect}_n(X)/\mathrm{iso}$$

given by mapping $[\phi] \in [X, \mathrm{Gr}_n]$ to $\phi^*(\gamma^n)$.

We have proved in the lectures that the map θ is well defined, and it remains to show that it is a bijection. To do this, we proceed by the following steps:

Exercise 1. Let $\xi = (\pi\colon E \to X) \in \mathrm{Vect}_n(X)$ be an n-bundle on X. Show that there is a bijection between

$$\{\text{bundle maps } \psi\colon E \to \underline{\mathbf{F}}^\infty_X, \text{ linear injection on each fiber}\}$$

and

$$\{\text{bundle isomorphisms } \xi \cong \phi^*(\gamma^n) \text{ for some } \phi\colon X \to \mathrm{Gr}_n\}$$

Exercise 2 (Surjectivity of θ)**.** Let $\xi = (\pi\colon E \to X) \in \mathrm{Vect}_n(X)$ be an n dimensional vector bundle on X. Show that there is a map $\psi\colon E \to \underline{\mathbf{F}}^\infty_X$ which is a linear injection in each fiber. Use Exercise 1 to conclude that θ is surjective.

We must now show that θ is injective. So suppose that $[\phi_0], [\phi_1] \in [X, \mathrm{Gr}_n]$ are such that $\theta([\phi_0]) = \theta([\phi_1])$, i.e., $\phi_0^*(\gamma^n) \cong \phi_1^*(\gamma^n)$. We wish to show that ϕ_0 and ϕ_1 are homotopic maps.

Let E denote the total space of the vector bundle $\phi_0^*(\gamma^n)$ on X. By Exercise 1, the isomorphism $\phi_0^*(\gamma^n) \cong \phi_1^*(\gamma^n)$ gives rise to two maps

$$\psi_0, \psi_1\colon E \to \underline{\mathbf{F}}^\infty_X.$$

Exercise 3. Argue that in order to show that $\phi_0 \simeq \phi_1$, it suffices to prove that $\psi_0 \simeq \psi_1$.

Exercise 4. Define two homotopies

$$F_t, G_t\colon \mathbf{F}^\infty \to \mathbf{F}^\infty$$

by

$$F_t(x_1, x_2, \dots) := (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots)$$

and

$$G_t(x_1, x_2, \dots) := (1-t)(x_1, x_2, \dots) + t(0, x_1, 0, x_2, 0, \dots).$$

Show that F_t and G_t are linear injections and use these to define a homotopy $\psi_0 \simeq \psi_1$.

Exercise 5. Read §1.5 in the notes.