Exercises 4.

1) Suppose first we are given an isomorphism $\xi \cong \varphi^*(\gamma^n)$ for some $\varphi : X \longrightarrow Gr_n$. Then we have a commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "\cong"] \arrow[dr, "\pi"'] & \varphi^*(\gamma^n) \arrow[r, "\varphi'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \gamma^n \arrow[r, "\psi^n"] \arrow[d] & \mathbb{F}^\infty_X \\
& X \arrow[r, "\varphi"] & Gr_n &
\end{tikzcd}$$

where $\psi^n((P, v)) := v$.
Then the composition

$$\psi := \psi^n \circ \varphi' : E \longrightarrow \mathbb{F}^\infty_X$$

is a linear injection on each fiber.

Conversely, suppose we are given a map $\psi : E \longrightarrow \mathbb{F}^\infty_X$. Define a map

$$\varphi : X \to Gr_n$$

by $\varphi(x) := \psi(\pi^{-1}(x)) \in Gr_n$.
Then we obtain a commutative diagram as above along with an isomorphism $E \cong \varphi^*(\gamma^n)$.

2) We know that E embeds in a trivial bundle, which gives the map

$$\psi : E \to \underline{\mathbb{F}}^\infty_X .$$

By 1) this map corresponds to a map $\varphi : X \to Gr_n$ such that $\varphi^*(\gamma^n) \cong E$. In other words, $\Theta([\varphi]) = E$.

3) Suppose $\psi_0 \overset{\psi_t}{\simeq} \psi_1$. Define $\varphi_t : X \to Gr_n$ by

$$\varphi_t(x) := \psi_t(\pi^{-1}(x)).$$

This gives a homotopy from φ_0 to φ_1.

4) It's clear that F_t & G_t are linear injections. F_t gives a homotopy

$$\psi_0 = F_0 \circ \psi_0 \simeq F_1 \circ \psi_0 =: \tilde{\psi}_0 ,$$

and similarly G_t gives a homotopy

$$\psi_1 = G_0 \circ \psi_1 \simeq G_1 \circ \psi_1 =: \tilde{\psi}_1 .$$

Let

$$\tilde{\psi}_t := (1-t)\,\tilde{\psi}_0 + t\,\tilde{\psi}_1 .$$

Then we have homotopies

$$\psi_0 \overset{F_t}{\simeq} \tilde{\psi}_0 \overset{\tilde{\psi}_t}{\simeq} \tilde{\psi}_1 \overset{G_t}{\simeq} \psi_1 .$$