

Here are solutions for the exercises we did not discuss in class on 06.11.:

Exercise. For a connected space X with universal covering $\tilde{X}$, prove that the $\pi_1(X, x_0)$ -action on $\pi_n(X, x_0) \simeq \pi_n(\tilde{X}, x_0)$ can be recovered from the action of $\pi_1(X, x_0)$ by deck transformations. More precisely,

$$\gamma p_*(\alpha) = p_*(\beta_{\tilde{\gamma}}(d_{\gamma,*}(\alpha))),$$

where $\tilde{\gamma}$ is the path in $\tilde{X}$ obtained by lifting γ with start point $\tilde{x}_0$, $\beta_{\tilde{\gamma}}$ denotes the associated isomorphism $\pi_n(X, \tilde{\gamma}(1)) \rightarrow \pi_n(X, \tilde{x}_0)$ and d_{γ} is the deck transformation $\tilde{X} \rightarrow \tilde{X}$ associated to γ .

Proof. For $\alpha : S^n \rightarrow \tilde{X}$, $d_{\gamma,*}(\alpha)$ is the unique lift of $p \circ \alpha$ to a map $S^n \rightarrow \tilde{X}$ taking the basepoint $p_0 \in S^n$ to $\tilde{\gamma}(1)$. Attaching $\tilde{\gamma}$ on the outside as in the definition of $\beta_{\tilde{\gamma}}$, we obtain a new map $S^n \rightarrow \tilde{X}$ taking the basepoint $p_0 \mapsto \tilde{x}_0$. This lifts (not just up to homotopy, but exactly!) the result of attaching γ on the outside of $p \circ \alpha$, which represents of $\gamma p_*(\alpha)$. This means the equality we want to prove holds exactly, finishing the proof.

For a bit of context: Since $\tilde{X}$ is simply-connected, any point in $\tilde{X}$ can be connected to $\tilde{x}_0$ by a path which is unique up to homotopy. As a result, it is not difficult to prove that the canonical map $\pi_n(\tilde{X}, \tilde{x}_0) \rightarrow [S^n, \tilde{X}]$ is a bijection, where the latter denotes the set of homotopy classes of *unpointed* maps $S^n \rightarrow \tilde{X}$ up to homotopy. This is a priori only a set, but carries an obvious action by deck transformations. Since under that bijection $d_{\gamma,*}(\alpha) \in [S^n, \tilde{X}]$ corresponds to the pointed map $\beta_{\tilde{\gamma}}(d_{\gamma,*}(\alpha))$, the above shows precisely that this action by deck transformations recovers the previously defined $\pi_1(X)$ -action on $\pi_n(X)$. $\square$

Exercise. For a map $f : S^3 \rightarrow S^2 \vee S^2$, the mapping cone $C(f)$ admits canonical basis elements $\beta \in H^4(C(f))$ and $\alpha_1, \alpha_2 \in H^2(C(f))$ from the connecting homomorphism $\delta : H^3(S^3) \rightarrow H^4(C(f))$ and the restriction map $H^2(C(f)) \rightarrow H^2(S^2 \vee S^2)$. Define $\ell(f) \in \mathbb{Z}$ as the unique integer such that

$$\alpha_1 \cup \alpha_2 = \ell(f)\beta \text{ in } H^*(C(f)).$$

Prove the following:

1. $\ell : \pi_3(S^2 \vee S^2) \rightarrow \mathbb{Z}$ is a homomorphism. (Proceed similarly to the proof of additivity of the Hopf invariant in the lecture.)
2. $\ell(f) = 1$ if $f : S^3 \rightarrow S^2 \vee S^2$ is the attaching map of the 4-cell in the canonical cell structure on $S^2 \times S^2$ with one 0-cell, two 2-cells and one 4-cell.
3. ℓ vanishes on anything in the image of the map $\pi_3(S^2) \oplus \pi_3(S^2) \rightarrow \pi_3(S^2 \vee S^2)$ induced by the inclusion maps of the two wedge summands.

Proof. For $f, g : S^3 \rightarrow S^2 \vee S^2$, $f+g$ is the composite $S^3 \xrightarrow{c} S^3 \vee S^3 \xrightarrow{f \vee g} S^2 \vee S^2$, where c is the map which collapses the equator. We have a corresponding

diagram

$$\begin{array}{ccccc}
S^3 & \xrightarrow{c} & S^3 \vee S^3 & \xleftarrow{i_1} & S^3 \\
\downarrow f+g & & \downarrow f \vee g & & \downarrow f \\
S^2 \vee S^2 & \xrightarrow{=} & S^2 \vee S^2 & \xleftarrow{=} & S^2 \vee S^2 \\
\downarrow & & \downarrow & & \downarrow \\
C(f+g) & \longrightarrow & C(f \vee g) & \longleftarrow & C(f)
\end{array}$$

The columns give long exact sequences on cohomology, and we obtain that $H^4(C(f \vee g)) \cong H^3(S^3 \vee S^3) \cong \mathbb{Z}\{\beta_1, \beta_2\}$, $H^2(C(f \vee g)) \cong \mathbb{Z}\{\alpha_1, \alpha_2\}$, where β_i and α_i denote the canonical generators coming from cohomology of spheres. Similarly, $H^4(C(f+g)) \cong \mathbb{Z} \cdot \beta$ and $H^2(C(f+g)) \cong \mathbb{Z} \cdot \alpha_1, \alpha_2$, analogously for $C(f)$ and $C(g)$. The left horizontal maps take $\beta_1, \beta_2 \mapsto \beta$, whereas the right horizontal map to $H^*(C(f))$ takes $\beta_1 \mapsto \beta$, $\beta_2 \mapsto 0$. For $C(g)$ (not in the diagram, but analogous), we have $\beta_2 \mapsto \beta$ and $\beta_1 \mapsto 0$. In $H^*(C(f))$, we have

$$\alpha_1 \alpha_2 = \ell(f) \beta,$$

analogously for $C(g)$. Comparing with the map induced by $C(f) \rightarrow C(f \vee g)$ and $C(g) \rightarrow C(f \vee g)$, we learn that in $H^*(C(f \vee g))$,

$$\alpha_1 \alpha_2 = \ell(f) \beta_1 + \ell(g) \beta_2.$$

Under the map induced by $C(f+g) \rightarrow C(f \vee g)$, this relation goes to

$$\alpha_1 \alpha_2 = (\ell(f) + \ell(g)) \beta.$$

So $\ell(f+g) = \ell(f) + \ell(g)$, proving the first part.

For the second part, we simply observe that in $H^*(S^2 \times S^2) \cong H^*(S^2) \otimes H^*(S^2)$ a generator of H^4 is given by $\alpha_1 \alpha_2$. Since $C(f) \cong S^2 \times S^2$ here, this implies $\ell(f) = 1$ (maybe ± 1 if we don't trust that the conventions of picking an orientation for spheres work out).

For the final statement, we may use that ℓ is a homomorphism to reduce to the case of an element coming from $\pi_3(S^2)$ along either of the two inclusions (since an element of the direct sum is a sum of two such elements). Without limiting generality we may assume it comes from the first summand, so we have a composite $S^3 \xrightarrow{f} S^2 \xrightarrow{i_1} S^2 \vee S^2$. Then $C(i_1 \circ f) = C(f) \vee S^2$. The map $C(f) \rightarrow C(i_1 \circ f)$ induces an isomorphism on H^4 , but on H^2 takes $\alpha_2 \mapsto 0$. In particular, it takes $\alpha_1 \alpha_2 \mapsto 0$, so $\alpha_1 \alpha_2 = 0$, and thus $\ell(f) = 0$. $\square$