

Solution (Hatcher Section 3.1, Exercise 5).

- (a) We define a singular 2-simplex $\sigma : \Delta^2 \rightarrow X$ whose faces are g , $f.g$ and f . If we view the standard 2-simplex as set of $(x_0, x_1, x_2) \in \mathbb{R}^3$ with $\sum x_i = 1$, $x_i \geq 0$, or abstractly as set of convex linear combinations $\sum x_i v_i$ with $\sum x_i = 1$ and $x_i \geq 0$ (where v_i are the vertices of Δ^2), then we may write σ as:

$$\sigma(x_0, x_1, x_2) = \begin{cases} f(2t(x_0, x_1, x_2)) & \text{if } t(x_0, x_1, x_2) < \frac{1}{2} \\ g(2t(x_0, x_1, x_2) - 1) & \text{if } t(x_0, x_1, x_2) \geq \frac{1}{2} \end{cases}$$

where $t(x_0, x_1, x_2) = \frac{1}{2}x_1 + x_2$ is designed to be linear and take the values 0, $\frac{1}{2}$ and 1 on the vertices v_0, v_1, v_2 . So along the face $[v_0, v_1]$, t goes from 0 to $\frac{1}{2}$, and thus we have $\sigma|_{[v_0, v_1]} = f$, while $\sigma|_{[v_1, v_2]}$ analogously. Finally, along $[v_0, v_2]$, t goes from 0 to 1 and we recover $\sigma|_{[v_0, v_2]} = f.g$. So:

$$0 = \delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(g) - \varphi(f.g) + \varphi(f)$$

and thus $\varphi(f.g) = \varphi(f) + \varphi(g)$.

- (b) For a constant path f , we have $f.f = f$, and so

$$\varphi(f) = \varphi(f.f) = \varphi(f) + \varphi(f),$$

from which we get $\varphi(f) = 0$.

- (c) Given a homotopy $H : I \times I \rightarrow X$ of f and g relative endpoints, we may proceed in two different ways. The first is to observe that H is constant along $\{0\} \times I$ and $\{1\} \times I$. So H factors through a map $H' : I \times I / \{0\} \times I \rightarrow X$. Here $I \times I / \{0\} \times I$ is homeomorphic to a triangle, and so we can use such a homeomorphism to produce a singular simplex $\sigma : \Delta^2 \rightarrow X$ whose faces are f , g and a constant path. Then

$$\delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(f) - \varphi(g) + \varphi(\text{const}) = \varphi(f) - \varphi(g),$$

and so $\varphi(f) = \varphi(g)$. The details are a little bit annoying. A somewhat more systematic approach is to observe that the square $I \times I$ can be triangulated into two triangles: One with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, and one with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. Let σ_1 and σ_2 be the two maps $\Delta^2 \rightarrow X$ we get by restricting to those simplices. Writing γ for the path which we obtain by restricting H to the diagonal of $I \times I$, we get

$$\begin{aligned} 0 &= \delta\varphi(\sigma_1) = \varphi(\partial\sigma_1) = \varphi(g) - \varphi(\gamma) + \varphi(\text{const}) \\ 0 &= \delta\varphi(\sigma_2) = \varphi(\partial\sigma_2) = \varphi(\text{const}) - \varphi(\gamma) + \varphi(g) \end{aligned}$$

which we may combine to again learn $\varphi(f) = \varphi(g)$.

(d) If φ is a coboundary, $\varphi = \delta\psi$, we have

$$\varphi(f) = \delta\psi(f) = \psi(\partial(f)) = \psi(f(1)) - \psi(f(0)),$$

which evidently depends only on the endpoints. Conversely, assume that $\varphi(f)$ only depends on the endpoints of f . For every path-connected component of X , we choose a representative $x_i \in X$. We then define

$$\psi(x) = \varphi(f), \text{ where } f : I \rightarrow X \text{ is a path with } f(1) = x \text{ and } f(0) = x_i,$$

x_i of course the representative of whichever path-connected component x lies in. By assumption, this is well-defined, i.e. does not depend on the choice of path f . We now need to prove $\varphi = \delta\psi$. For this, let g be an arbitrary path with endpoints $x' = g(1)$ and $x = g(0)$, and f a path from $f(0) = x_i$ and $f(1) = x$. Then $f.g$ is a path from x_i to x' , so we have

$$\begin{aligned} \psi(x') &= \varphi(f.g) = \varphi(f) + \varphi(g), \\ \psi(x) &= \varphi(f) \end{aligned}$$

It follows that

$$\delta\psi(g) = \psi(x') - \psi(x) = \varphi(g),$$

as desired.

Solution (Hatcher page 193, Exercise). We need to show that if $A \rightarrow B \rightarrow C \rightarrow 0$ is exact, then also $0 \rightarrow C^* \rightarrow B^* \rightarrow A^*$ is exact, where $A^* = \text{Hom}(A, G)$ etc., for a fixed abelian group G . Clearly the compositions are 0. To check exactness on the left, we need to prove that any $C \rightarrow G$ for which the composite $B \rightarrow C \rightarrow G$ is zero, was zero to begin with. But this is clear from surjectivity of $B \rightarrow C$. For exactness in the middle, we need to show that any $B \rightarrow G$ for which $A \rightarrow B \rightarrow G$ is zero, factors through $C \rightarrow G$. But this is clear from the fact that $B \rightarrow C$ induces an isomorphism $B/\text{im}(A \rightarrow B) \rightarrow C$, and any morphism $B \rightarrow G$ which vanishes on the image of $A \rightarrow B$ factors through that quotient.