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Homotopy groups

Homotopy groups can be thought of as a higher dimensional analogue of the fundamental group $\pi_1(X, x_0)$, where the role of S^1 is replaced by S^n . Just as for the fundamental group, where definitions were simpler if done in terms of the unit interval $I = [0, 1]$, we will work with the n -dimensional cube I^n instead of S^n .

Definition. For a pointed space (X, x_0) , we let $\pi_n(X, x_0)$ denote the set of homotopy classes of maps of pairs

$$(I^n, \partial I^n) \rightarrow (X, x_0).$$

(Note that since $I^n/\partial I^n \cong S^n$, this is the same as pointed homotopy classes of pointed maps $S^n \rightarrow X$.)

For a pointed map $f : X \rightarrow Y$, postcomposing gives a map $\pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$. This makes π_n into a functor from Top_* (pointed topological spaces) to Set (but, as we will see in a second, even Grp , resp. Ab for $n \geq 2$).

Given two maps $f, g : (I^n, \partial I^n) \rightarrow (X, x_0)$, we may “stack” them to a single map $f + g$, via

$$(f + g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots) & \text{if } t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots) & \text{if } t_1 \geq \frac{1}{2} \end{cases}$$

For $n = 1$, this recovers path composition in the fundamental group. For the same reasons as given there, $+$ is associative, unital with respect to the constant map with value x_0 , and admits inverses given as

$$(-f)(t_1, \dots, t_n) = f(1 - t_1, \dots, t_n).$$

For $n \geq 2$, $+$ is also commutative, justifying the additive notation. Indeed, if we think of $f + g$ as two cubes shrunk and stacked in the first coordinate direction, we have a homotopy between $f + g$ and $g + f$ obtained by first shrinking the constituent cubes also in the second dimension, then moving them around each other, then expanding them again. This can of course be written in explicit formulas, but it is a bit tedious to do so. A slicker way to package this is as follows:

Lemma. $\pi_n(X, x_0)$ is a commutative group.

Proof. We define an alternative version of $+$ by using the second coordinate:

$$(f +_2 g)(t_1, t_2, \dots, t_n) = \begin{cases} f(t_1, 2t_2, \dots) & \text{if } t_2 \leq \frac{1}{2} \\ g(t_1, 2t_2 - 1, \dots) & \text{if } t_2 \geq \frac{1}{2} \end{cases}$$

Now given four elements a, b, c, d , we have

$$(a + b) +_2 (c + d) = (a +_2 c) + (b +_2 d)$$

(not just up to homotopy, as one sees by writing both sides out). Just as for $+$, it is easy to see that $+_2$ is unital up to homotopy with respect to the constant map which we now write as 0. Then:

$$f + g \simeq (f +_2 0) + (0 +_2 g) = (f + 0) +_2 (0 + g) \simeq f +_2 g,$$

but also

$$f + g \simeq (0 +_2 f) + (g +_2 0) = (0 + g) +_2 (f + 0) \simeq g +_2 f,$$

so altogether $f + g \simeq f +_2 g \simeq g + f$. This proves commutativity, and also that “stacking” in other coordinate directions would produce the same group structure on $\pi_n(X, x_0)$. $\square$

If we view elements in homotopy groups as pointed maps $S^n \rightarrow X$, then the above group structure can also be described as composite

$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X,$$

where c is the quotient map which collapses the equator. This perspective will be useful later.

Just as $\pi_1(X, x_0)$ does not depend on the choice of a basepoint, the same holds for higher homotopy groups:

Lemma. *Given $x_0, x_1 \in X$ in the same path connected component, we have an isomorphism*

$$\pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$$

Proof. Let γ denote a path between the two points. For $f : (I^n, \partial I^n) \rightarrow (X, x_1)$, we may attach the collar $\partial I^n \times I \xrightarrow{\gamma \circ p_2} X$ and use a fixed homeomorphism back to I^n to define $\gamma f : (I^n, \partial I^n) \rightarrow (X, x_0)$. Then the homotopy class of γf depends only on homotopy classes of γ and f , one has $(\gamma \cdot \eta)f \simeq \gamma(\eta f)$ and thus inverse paths give inverse bijections.

It is also straightforward to check that the map $f \mapsto \gamma f$ is a homomorphism $\pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$. $\square$

For this reason, it is often customary to shorten notation to $\pi_n(X)$, especially for connected spaces. Note however that while all $\pi_n(X, x_0)$ are isomorphic if X is connected, different homotopy classes of paths can give different isomorphisms! So especially if $\pi_1(X, x_0) \neq 0$, it can be important to keep track of basepoints.

Remark. *For a loop γ at the base-point $x_0 \in X$, the map $f \mapsto \gamma f$ is an automorphism of $\pi_n(X, x_0)$. This defines an action of the group $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$, through group automorphisms. In the case $n = 1$, this is the action of $\pi_1(X, x_0)$ on itself via conjugation, since $\gamma\eta = \gamma \cdot \eta \cdot \gamma^{-1}$.*

Some nontrivial examples

We now want to see some nontrivial examples of homotopy groups. We of course understand π_1 very well, through tools like covering theory and Seifert-van Kampen. One easy example is $\pi_1(S^1) = \mathbb{Z}$. What can we say about $\pi_n(S^n)$?

Definition. For an element $[f] \in \pi_n(S^n)$, we let $\deg(f) \in \mathbb{Z}$ denote the degree of f , i.e. the unique integer such that

$$f_* : H_n(S^n) \rightarrow H_n(S^n)$$

is multiplication by $\deg(f)$.

Lemma. $\deg : \pi_n(S^n) \rightarrow \mathbb{Z}$ is a surjective group homomorphism.

Proof. For $f, g : S^n \rightarrow S^n$ two pointed maps, their sum $f + g$ is given by the composite

$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} S^n,$$

which, on H_n , induces

$$\mathbb{Z} \xrightarrow{(1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(\deg(f), \deg(g))} \mathbb{Z}.$$

This composite is multiplication by $\deg(f) + \deg(g)$.

Surjectivity follows from the fact that $\deg(\text{id}_{S^n}) = 1$. □

It follows that $\pi_n(S^n)$ admits $\mathbb{Z}$ as a direct summand, i.e. it is at least as big as $\mathbb{Z}$. We will see later that it is exactly $\mathbb{Z}$, i.e. that the degree is a complete invariant of the homotopy class of a map $S^n \rightarrow S^n$.

For now, we have not seen anything to suggest that homotopy behaves differently from homology. We will next see that $\pi_3(S^2) \neq 0$, indicating that homotopy groups are much more subtle than homology groups.

Definition. Given a map $f : A \rightarrow X$, its mapping cone is the space $X \cup_f CA$, where $CA = A \times [0, 1]/A \times \{1\}$ is the cone on A , which we glue to X along f .

One example of mapping cones we are already familiar with is the following: If $\alpha : S^{n-1} \rightarrow X$, then $C(\alpha) = X \cup_\alpha D^n$ is the result of attaching a cell to X along α .

The mapping cone is a better version of X/A , not requiring A to be a nice subspace of X . For example, if $p \in CA$ denotes the tip of the cone, the long exact sequence of the pair $(C(f) \setminus \{p\}, A \times (0, 1))$ takes the form

$$\dots \rightarrow H_*(A \times (0, 1)) \rightarrow H_*(C(f) \setminus \{p\}) \rightarrow H_*(C(f) \setminus \{p\}, A \times (0, 1)) \rightarrow \dots$$

The first term is equivalent to $H_*(A)$, the second to $H_*(X)$, by deformation retractions. The third term agrees with the homology of $C(f)$ relative to the open cone by excision, and since the open cone deformation retracts to p , with reduced homology. So we have a long exact sequence

$$\dots \rightarrow H_*(A) \xrightarrow{f_*} H_*(X) \rightarrow \tilde{H}_*(C(f)) \rightarrow \dots,$$

where the identification of the map on the left comes from explicitly following the deformation retractions. There is also a version where all three terms are reduced, and one for cohomology, constructed analogously. If f is the inclusion of a good enough subspace, we have already seen a similar sequence with X/A instead of $C(f)$, this can be recovered from the version here by observing that in such a case the canonical map $C(f) \rightarrow X/A$ collapsing CA is a homotopy equivalence. So the version with $C(f)$ is a generalisation of that.

Lemma. *If $f, g : A \rightarrow X$ are homotopic, $C(f) \simeq C(g)$.*

Proof. Let $H : A \times I \rightarrow X$ be a homotopy. Then $C(H)$ deformation retracts onto both $C(f)$ and $C(g)$ (using that $C(A \times I)$ deformation retracts onto $A \times I \cup CA \times \{0\}$ and $A \times I \cup CA \times \{1\}$). $\square$

So we can use invariants of $C(f)$ to distinguish homotopy classes of f .

Example. $\mathbb{C}P^2$ has a cell structure with one cell each of dimensions 0, 2, 4. So we may write $\mathbb{C}P^2$ as $C(\eta)$ for η the attaching map $S^3 \rightarrow S^2$ of the 4-cell.

If $\eta \simeq 0$, we would therefore have a homotopy equivalence $\mathbb{C}P^2 \simeq C(0) = S^2 \vee S^4$. But in $\tilde{H}^*(S^2 \vee S^4)$, cup products are trivial, whereas in $\tilde{H}^*(\mathbb{C}P^2)$, they are not. So η is not homotopic to a constant map and $\pi_3(S^2) \neq 0$.

We want to understand this a little bit better. Note that for any map $f : S^3 \rightarrow S^2$, the long exact sequence gives isomorphisms $H^3(S^3) \rightarrow H^4(C(f))$ and $H^2(C(f)) \rightarrow H^2(S^2)$. So once we have fixed orientations of spheres, we get preferred generators $\alpha \in H^2(C(f))$ and $\beta \in H^4(C(f))$.

Definition. *The Hopf invariant $h(f) \in \mathbb{Z}$ is the unique number such that*

$$\alpha^2 = h(f)\beta$$

in $H^*(C(f))$.

Example. We have $h(\eta) = 1$, since that is how cup products in $H^*(\mathbb{C}P^2)$ work.

Lemma. *The map $h : \pi_3(S^2) \rightarrow \mathbb{Z}$ is a surjective homomorphism.*

Proof. If $f, g : S^3 \rightarrow S^2$ are two maps, we consider the following diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{c} & S^3 \vee S^3 & \xleftarrow{i_1} & S^3 \\ \downarrow f+g & & \downarrow f \vee g & & \downarrow f \\ S^2 & \xrightarrow{\text{id}} & S^2 & \xleftarrow{\text{id}} & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ C(f+g) & \longrightarrow & C(f \vee g) & \longleftarrow & C(f). \end{array} \quad \text{Each of the columns gives rise to a long}$$

exact sequence. The middle of those determines $H^*(C(f \vee g))$, with $H^2(C(f \vee g))$ free on α and $H^4(C(f \vee g))$ free on β_1, β_2 . The map to $H^*(C(f))$ takes $\alpha \mapsto \alpha$ and $\beta_1 \mapsto \beta, \beta_2 \mapsto 0$. The map to $H^*(C(g))$ (not in the picture) works analogously with β_1, β_2 interchanged. Together they show that $\alpha^2 = h(f)\beta_1 + h(g)\beta_2$ in $H^*(C(f \vee g))$. Now the induced map to $C(f+g)$ takes $\beta_1, \beta_2 \mapsto \beta$, so in $H^*(C(f+g))$ we have $\alpha^2 = (h(f) + h(g))\beta$. This shows additivity, and surjectivity is clear from the fact that $h(\eta) = 1$. $\square$

So $\pi_3(S^2)$ has $\mathbb{Z}$ as a direct summand.