1 Vector bundles

In the following, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Smooth manifolds are assumed to be second countable. For a smooth manifold $M$ we denote by $\mathcal{D}(M, \mathbb{K})$ the ring of smooth functions $M \to \mathbb{K}$, and we set $\mathcal{D}(M) := \mathcal{D}(M, \mathbb{R})$.

By definition, a smooth $\mathbb{K}$-vector bundle of rank $r$ over a smooth manifold $M$ consists of

- a smooth manifold $E$ called the total space of the bundle,
- a smooth map $\pi : E \to M$ called the projection,
- for each $p \in M$ a $\mathbb{K}$-vector space structure on the fibre $E_p := \pi^{-1}(p)$,

such that the axiom of local triviality holds: For each $p \in M$ there should exist an open neighbourhood $U$ of $p$ and a diffeomorphism $\Phi : U \times \mathbb{K}^r \to \pi^{-1}U$ such that

- $\pi \circ \Phi(q, v) = q$ for all $(q, v) \in U \times \mathbb{K}^r$,
- for each $q \in U$ the map
  \[ \Phi_q : \mathbb{K}^r \to E_q, \quad v \mapsto \Phi(q, v) \]
  is a linear isomorphism.

Such a diffeomorphism $\Phi$ is called a local trivialization of $E$. If $r = 1$ then $E$ is called a (real or complex) line bundle over $M$.

Examples

(i) The product bundle

$M \times V \to M, \quad (q, v) \mapsto q,$

where $V$ is any finite-dimensional $\mathbb{K}$-vector space.
(ii) The tangent bundle $TM \to M$.

(iii) If $\pi : E \to M$ is a vector bundle and $N \subset M$ a submanifold, then $E|_N := \pi^{-1}N$ is a vector bundle over $N$.

(iv) More generally, if $\pi : E \to M$ is a vector bundle and $f : N \to M$ a smooth map, then

$$f^*E := \{(q,v) \in N \times E \mid f(q) = \pi(v)\}$$

is a vector bundle over $N$ called the **pull-back** of $E$ by $f$.

(v) Many operations on vector spaces can be applied fibrewise on a vector bundle to produce new vector bundles. For instance, to any vector space $V$ we can associate its dual space $V^*$. Applying this operation on each fibre of a vector bundle $E \to M$ yields the **dual bundle** $E^* \to M$ whose fibre over a point $p \in M$ is $(E_p)^*$. In the case $E = TM$, the dual bundle $T^*M$ is called the **cotangent bundle** of $M$. More details about this type of construction will be given in Section 2.

Let $E \to M$ be a vector bundle of rank $r$. By a **subbundle** of $E$ of rank $s$ we mean a subset $E' \subset E$ such that for every point $p \in M$ there exist a local trivialization $\Phi : U \times \mathbb{K}^r \to \pi^{-1}U$ of $E$ around $p$ and an $s$–dimensional linear subspace $V \subset \mathbb{K}^r$ such that

$$\Phi(U \times V) = \pi^{-1}U \cap E'.$$

Then $E'$ is in a natural way a vector bundle over $M$.

**Example** If $N \subset M$ is a submanifold then the tangent bundle $TN$ is a subbundle of $TM|_N$.

By a **section** of a vector bundle $\pi : E \to M$ we mean a smooth map $s : M \to E$ such that $\pi \circ s = \text{Id}_M$. The vector space $\Gamma(E)$ of all sections of $E$ is a module over the ring $\mathcal{D}(M, \mathbb{K})$. Addition in $\Gamma(E)$ and multiplication by functions are defined pointwise by

$$(s + t)(p) := s(p) + t(p),$$

$$(fs)(p) := f(p) s(p)$$

for $s, t \in \Gamma(E)$ and $f \in \mathcal{D}(M, \mathbb{K})$.

By definition, a **vector field** on $M$ is a section of $TM$. We denote by $\mathcal{X}(M) = \Gamma(TM)$ the $\mathcal{D}(M)$–module of all vector fields on $M$. 

2
Let $\pi : E \to M$ and $\pi' : E' \to M'$ be $K$-vector bundles and $f : M \to M'$ a map. A smooth map $F : E \to E'$ is called a bundle homomorphism over $f$ if $F$ maps each fibre $E_p$ linearly into $E'_{f(p)}$. (This implies that $f$ is smooth.) If in addition $M = M'$ and $f$ is the identity map then $F$ is called a bundle homomorphism. A bundle homomorphism $F : E \to E'$ which is also a diffeomorphism is called a bundle isomorphism. A vector bundle is called trivial if it is isomorphic to a product bundle. An isomorphism $E \to E$ is called a bundle automorphism or gauge transformation.

**Examples**

(i) If $f : M \to M'$ is a smooth map between manifolds then the tangent map $Tf : TM \to TM'$ is a bundle homomorphism over $f$.

(ii) If $\pi : E \to M$ is a vector bundle and $f : N \to M$ a smooth map then there is a canonical bundle homomorphism

$$f^* E \to E$$

obtained by restricting the projection $N \times E \to E$ to $f^* E$.

(iii) Let $E = M \times K^r$ and $E = M \times K^s$ be product bundles over $M$. Then a bundle homomorphism $E \to E'$ has the form

$$M \times K^r \to M \times K^s, \quad (p, v) \mapsto (p, \alpha(p)v)$$

for some smooth map $\alpha$ from $M$ into the space $M(s \times r, K)$ of $s \times r$ matrices with entries in $K$.

Let $E \to M$ be a $K$-bundle of rank $r$ and let $U \subset M$ be an open subset. By a (local) frame for $E$ over $U$ we mean an $r$-tuple $(s_1, \ldots, s_r)$ of sections of $E|_U$ such that $(s_1(p), \ldots, s_r(p))$ is a basis for $E_p$ for all $p \in U$. If $U = M$ then $(s_1, \ldots, s_r)$ is called a global frame. In that case the map

$$M \times K^r \to E, \quad (p, v) \mapsto \sum_{j=1}^r v^j s_j(p)$$

is a bijective bundle homomorphism and therefore an isomorphism. Conversely, such an isomorphism clearly gives rise to a global frame.

## 2 Operations on vector bundles

Let $C$ be the category of finite-dimensional $K$–vector spaces and linear isomorphisms, and let $\mathcal{F} : C \to C$ be a covariant functor. In other words, $\mathcal{F}$ is a rule that associates to any vector space $V$ a vector space $\mathcal{F}(V)$ and to any isomorphism $A : V \to W$ between vector spaces an isomorphism $\mathcal{F}(A) : \mathcal{F}(V) \to \mathcal{F}(W)$, such that the following hold:
• If \( V \xrightarrow{A} W \xrightarrow{B} X \) are isomorphisms then \( \mathcal{F}(B \circ A) = \mathcal{F}(B) \circ \mathcal{F}(A) \).

• \( \mathcal{F}(\text{Id}_V) = \text{Id}_{\mathcal{F}(V)} \).

Such a functor is called smooth if for any vector space \( V \) the map

\[
\text{GL}(V) \to \text{GL}(\mathcal{F}(V)), \quad A \mapsto \mathcal{F}(A)
\]

is smooth (and therefore a homomorphism of Lie groups).

**Example** Given a non-negative integer \( k \), let \( \mathcal{F}(V) := \Lambda^k V^* \) be the vector space of alternating \( k \)-forms on \( V \). If \( A : V \to W \) is an isomorphism let \( A^* : W^* \to V^* \) be the dual isomorphism and set

\[
\mathcal{F}(A) := (A^*)^{-1} = (A^{-1})^*.
\]

Given a smooth functor \( \mathcal{F} : \mathcal{C} \to \mathcal{C} \) and a vector bundle \( \pi : E \to M \) we construct a new vector bundle \( \tilde{\pi} : \tilde{E} \to M \) as follows. (We will sometimes write \( \mathcal{F}(E) := \tilde{E} \).) As a set,

\[
\tilde{E} := \bigcup_{p \in M} \{ p \} \times \mathcal{F}(E_p),
\]

and \( \tilde{\pi}(p, w) := p \). For any local trivialization

\[
\Phi : U \times \mathbb{K}^r \to \pi^{-1}U
\]

the map

\[
\tilde{\Phi} : U \times \mathcal{F}(\mathbb{K}^r) \to \tilde{\pi}^{-1}U, \quad (p, w) \mapsto (p, \mathcal{F}(\Phi_p)w)
\]

is a bijection. Moreover, if \( \Psi \) is another local trivialization of \( E \) defined over an open subset \( V \subset M \) then

\[
\tilde{\Psi}^{-1}\tilde{\Phi}(p, w) = (p, \mathcal{F}(\Psi_p^{-1}\Phi_p)w)
\]

is an automorphism of the product bundle \( (U \cap V) \times \mathcal{F}(\mathbb{K}^r) \) over \( U \cap V \). Hence, \( \tilde{E} \) has a unique structure of vector bundle over \( M \) with projection \( \tilde{\pi} \) such that each map \( \tilde{\Phi} \) defines a local trivialization of \( \tilde{E} \) (after fixing a basis for \( \mathcal{F}(\mathbb{K}^r) \)).

**Examples**

(i) Taking \( \mathcal{F}(V) := \Lambda^k V^* \) we obtain a vector bundle \( \Lambda^k E^* \). Similarly, taking \( \mathcal{F}(V) \) to be the whole exterior algebra \( \Lambda(V) := \sum_{k \geq 0} \Lambda^k V^* \) yields a vector bundle (in fact, bundle of real algebras) \( \Lambda E^* \) which contains each \( \Lambda^k E^* \) as a subbundle.
For $E = TM$, the bundles $\Lambda^k E^*$ and $\Lambda E^*$ will be denoted by $\Lambda^k M$ and $\Lambda M$, respectively. Note that the sections of $\Lambda^k M$ are by definition the differential $k$–forms on $M$.

The above construction can also be applied to functors of several variables. For instance, to any pair $V, W$ of vector spaces we can associate the direct sum $V \oplus W$, the tensor product $V \otimes W$, and the space of homomorphisms $\text{Hom}(V, W)$ (the latter being isomorphic to $V^* \otimes W$). Applying these functors fibrewise to a pair $E, E'$ of vector bundles over $M$ we obtain new vector bundles $E \oplus E'$, $E \otimes E'$, and $\text{Hom}(E, E')$ over $M$. The fibre of $E \oplus E'$ over a point $p \in M$ is $E_p \oplus E'_p$, and similarly for the other examples. In the case $E' = E$ we obtain the endomorphism bundle $\text{End}(E) := \text{Hom}(E, E)$.

Sections of the bundle $\Lambda^k M \otimes E$ are called differential $k$–forms on $M$ with values in $E$. The space of all such forms is denoted by $\Omega^k(M; E)$, or by $\Omega^k(M; V)$ in the case of a product bundle $E = M \times V$.

## 3 Connections

Given a section $s$ of a vector bundle $E \to M$ and a vector field $X$ on $M$ we would like to have some kind of derivative $\nabla_X s$ of $s$ with respect to $X$. This derivative should be a new section of $E$. Because there is no canonical isomorphism between the fibres of $E$ at two different points, we are unable to define a canonical derivative of this kind. Instead, we will formulate some properties that we would like such a derivative to have.

By a connection (or covariant derivative) in a $\mathbb{K}$-vector bundle $E \to M$ we mean a map

$$\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$

such that for all $X, Y \in \mathcal{X}(M)$, $s, t \in \Gamma(E)$, and $f \in \mathcal{D}(M, \mathbb{K})$ one has

(i) $\nabla_{X+Y}(s) = \nabla_X s + \nabla_Y s$,

(ii) $\nabla_{fX}(s) = f\nabla_X s$,

(iii) $\nabla_X(s + t) = \nabla_X s + \nabla_X t$,

(iv) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note that if $f$ is constant, say $f \equiv \alpha$, then $Xf = 0$, so (iv) gives

$$\nabla_X(\alpha s) = \alpha \nabla_X s.$$
A section $s$ is called **covariantly constant**, or **parallel**, if $\nabla_X s = 0$ for all vector fields $X$.

**Example** A product bundle $E = M \times V \to M$ has a canonical connection $\nabla$ called the **product connection**. To define this, note that to any function $h : M \to V$ we can associate a section $\tilde{h}$ of $E$ given by

$$\tilde{h}(p) = (p, h(p)),$$

and any section of $E$ has this form. Now define $\nabla_X \tilde{h}$ to be the section corresponding to the function $Xh$, i.e.

$$\nabla_X \tilde{h} := \tilde{Xh}.$$

**Theorem 3.1** Any vector bundle $E \to M$ admits a connection $\nabla$.

*Proof.* Choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of $M$ such that $E|_{U_\alpha}$ is trivial for each $\alpha$. Choose a partition of unity $\{g_\alpha\}$ subordinate to this cover. By the example above, each bundle $E|_{U_\alpha}$ admits a connection $\nabla^\alpha$. Now define, for any section $s$ of $E$ and vector field $X$ on $M$,

$$\nabla_X s := \sum_\alpha g_\alpha \nabla_X^\alpha (s),$$

where $\nabla^\alpha_X (s)$, defined initially on $U_\alpha$, is extended trivially to all of $M$. Then $\nabla$ clearly satisfies the first three axioms for a connection. To verify the fourth one, let $f \in \mathcal{D}(M, \mathbb{K})$. Then

$$\nabla_X (fs) = \sum_\alpha g_\alpha (Xf \cdot s + f \nabla^\alpha_X (s))$$

$$= Xf \cdot s + \sum_\alpha g_\alpha \nabla^\alpha_X (s)$$

$$= Xf \cdot s + f \nabla_X (s).$$

For an arbitrary connection $\nabla$ we will now investigate the dependence of $\nabla_X s$ on the two variables $X$ and $s$, beginning with $s$.

**Proposition 3.1** Let $\nabla$ be a connection in $E \to M$ and $X$ a vector field on $M$. Then $\nabla_X$ is a local operator, i.e. if a section $s$ of $E$ vanishes on an open subset $U \subset M$ then $\nabla_X s$ vanishes on $U$, too.
Proof. Suppose $s|_U = 0$ and let $p \in U$. Choose a smooth real function $f$ on $M$ which is supported in $U$ and satisfies $f(p) = 1$. Then $fs = 0$, so

$$0 = \nabla_X(fs) = Xf \cdot s + f \nabla_X s.$$ 

Evaluating this equation at $p$ gives $(\nabla_X s)(p) = 0$. □

**Proposition 3.2** Let $E, E'$ be $\mathbb{K}$-vector bundles over $M$ and

$$A : \Gamma(E) \rightarrow \Gamma(E')$$

a $\mathcal{D}(M, \mathbb{K})$–linear map. Then there exists a unique bundle homomorphism $a : E \rightarrow E'$ such that $As = as$ for all $s \in \Gamma(E)$.

Explicitly, the assumption on $A$ is that

$$A(s + t) = As + At, \quad A(fs) = fAs$$

for all $s,t \in \Gamma(E)$ and $f \in \mathcal{D}(M, \mathbb{K})$. The main point in the proposition is that $(As)(p)$ depends only on the value of $s$ at $p$.

**Proof of proposition.** Uniqueness follows from the fact that for any $p \in M$ and $v \in E_p$ there exists a section $s$ of $E$ with $s(p) = v$. To prove existence of $a$, let $p \in M$ and $v \in E_p$. Choose a section $s$ of $E$ with $s(p) = v$ and define

$$av := (As)(p).$$

To show that $av$ is independent of the choice of $s$ it suffices to verify that if $t$ is any section of $E$ with $t(p) = 0$ then $(At)(p) = 0$. Let $r$ be the rank of $E$. Choose sections $s_1, \ldots, s_r$ of $E$ which are linearly independent at every point in some neighbourhood $U$ of $p$. Choose a smooth real function $f$ on $M$ which is supported in $U$ and satisfies $f(p) = 1$. Then

$$ft = \sum_j g^j s_j$$

for some uniquely determined $\mathbb{K}$–valued functions $g^1, \ldots, g^r$ on $M$ that vanish outside $U$. Clearly, $g^j(p) = 0$. Applying $A$ to both sides of the above equation we obtain

$$fAt = \sum_j g^j As_j,$$

and evaluating both sides of this equation at $p$ gives $(At)(p) = 0$ as required. □
**Corollary 3.1** Let \( \nabla \) be a connection in \( E \to M \) and \( s \in \Gamma(E) \), \( p \in M \). If \( X, Y \) are vector fields on \( M \) satisfying \( X_p = Y_p \) then \( (\nabla_X s)(p) = (\nabla_Y s)(p) \).

**Proof.** This follows from the proposition because for given \( s \) the map

\[
X \mapsto \nabla_X s
\]

is \( \mathcal{D}(M) \)-linear. \( \square \)

The corollary allows us to define \( \nabla_v s \) for any tangent vector \( v \in T_pM \). Namely, choose any vector field \( X \) with \( X_p = v \) and set

\[
\nabla_v s := (\nabla_X s)(p).
\]

**Proposition 3.3** Connections in \( E \) are in 1-1 correspondence with \( \mathbb{R} \)-linear maps

\[
\bar{\nabla} : \Omega^0(M; E) \to \Omega^1(M; E)
\]

satisfying

\[
\bar{\nabla}(fs) = df \otimes s + f \nabla s
\]

for all \( f \in \mathcal{D}(M) \) and \( s \in \Gamma(E) \). If \( \bar{\nabla} \) is such a map then the corresponding connection \( \nabla \) is given by

\[
\nabla_X s = (\bar{\nabla}s)(X).
\]  

**Proof.** It is clear that (1) defines a connection in \( E \). To reconstruct \( \bar{\nabla} \) from \( \nabla \), let \( X_1, \ldots, X_n \) be a frame of vector fields on an open subset \( U \subset M \), and let \( \alpha^1, \ldots, \alpha^n \) the dual frame of 1–forms. Then

\[
(\bar{\nabla}s)_U = \sum_i \alpha^i \otimes \nabla_{X_i}(s). \quad \square
\]

Henceforth we will not distinguish between \( \nabla \) and \( \bar{\nabla} \).

For any vector bundle \( E \) let \( \mathcal{A}(E) \) denote the set of all connections in \( E \). The following proposition says that \( \mathcal{A}(E) \) is an affine space.

**Theorem 3.2** Given a connection \( \nabla \) in a vector bundle \( E \to M \), the map

\[
\Omega^1(M; \text{End}(E)) \to \mathcal{A}(E), \quad a \mapsto \nabla + a
\]

is a bijection.
Proof. It is easy to see that $\nabla + a$ is a connection. Conversely, if $\nabla'$ is any other connection in $E$ then $\nabla' - \nabla$ is $\mathcal{D}(M, \mathbb{K})$–linear and therefore given by a bundle homomorphism $E \to T^*M \otimes E$, or equivalently, a section of the bundle $E^* \otimes T^*M \otimes E = T^*M \otimes \text{End}(E)$. \qed

**Example** Let $E \to M$ be a trivial vector bundle and $(s_1, \ldots, s_r)$ a global frame for $E$. Then there is a 1-1 correspondence between connections in $E$ and $r \times r$ matrices $\omega = (\omega^i_j)$ of $\mathbb{K}$–valued 1–forms on $M$ specified by the formula

$$\nabla_X s_j = \sum_{i=1}^{r} \omega^i_j(X)s_i$$

for vector fields $X$ on $M$. To deduce this from the theorem, let $\nabla^0$ be the product connection in $E$ given by $\nabla_X s_j = 0$. Then any other connection in $E$ has the form $\nabla^0 + \omega$, where

$$\omega \in \Omega^1(M; \text{gl}(r, \mathbb{K})).$$

We call $\omega$ the **connection form** of $\nabla$ with respect to the given global frame.

**Proposition 3.4** Let $E_1, \ldots, E_m, E'$ be $\mathbb{K}$–vector bundles over $M$ and

$$B : \Gamma(E_1) \times \cdots \times \Gamma(E_m) \to \Gamma(E')$$

a $\mathcal{D}(M, \mathbb{K})$–multilinear map. Then there exists for each $p \in M$ a $\mathbb{K}$–multilinear map

$$B_p : (E_1)_p \times \cdots \times (E_m)_p \to (E')_p$$

such that if $s_j \in \Gamma(E_j)$, $j = 1, \ldots, m$ then

$$B(s_1, \ldots, s_m)(p) = B_p(s_1(p), \ldots, s_m(p)).$$

All these maps $B_p$ together define a bundle homomorphism $E_1 \otimes \cdots \otimes E_m \to E'$.

*Proof.* If at least one of the sections $s_j$ vanishes at $p$ then, by Proposition 3.2, $B(s_1, \ldots, s_m)$ also vanishes at $p$. By repeated application of this we see that if for each $j$ we are given a pair of sections $s_j, t_j$ of $E_j$ with the same value at $p$ then

$$B(s_1, \ldots, s_m)(p) = B(t_1, \ldots, t_m)(p).$$

We can now define $B_p$ by (3). \qed
If $\nabla$ is a connection in $TM$ then the map

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M), \quad (X,Y) \mapsto \nabla_X Y - \nabla_Y X - [X,Y]$$

is easily seen to be $\mathcal{D}(M)$–linear. Since it is also skew-symmetric, it is given by a $TM$–valued 2–form $T \in \Omega^2(M; TM)$ called the torsion of $\nabla$. If $T = 0$ then $\nabla$ is called torsion-free. For instance, the formula for the Lie-bracket in local coordinates says that the product connection in $\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is torsion-free. Any smooth manifold admits a torsion-free connection in $TM$, as follows from Theorems 8.1 and 8.4 below. (Alternatively, one can prove this by pathing together local torsion-free connections using a partition of unity.)

4 Pull-back connections

Let $\pi : E \to M$ be a vector bundle and $f : \tilde{M} \to M$ a smooth map. By a section of $E$ along $f$ we mean a smooth map $t : \tilde{M} \to E$ such that $\pi \circ t = f$. We will usually identify such a map $t$ with the corresponding section $\tilde{s}$ of the pull-back bundle $f^*E$ given by $\tilde{s}(p) = (p, t(p))$. Note that if $s$ is a section of $E$ then $f^*s := s \circ f$ is a section of $E$ along $f$.

**Proposition 4.1** Let $\nabla$ be a connection in $E \to M$ and $f : \tilde{M} \to M$ a smooth map. Then there exists a unique connection $\tilde{\nabla} = f^*\nabla$ in $\tilde{E} := f^*E$ such that for all $s \in \Gamma(E)$, $p \in \tilde{M}$, and $v \in T_p\tilde{M}$ we have

$$\tilde{\nabla}_v (f^*s) = \nabla_{f_*v}(s) \quad \text{in} \quad \tilde{E}_p = E_{f(p)},$$

where $f_* : T_p\tilde{M} \to T_{f(p)}M$ is the tangent map of $f$.

**Proof.** Uniqueness follows from the fact that if $\{s_j\}$ is a frame for $E$ over an open set $U \subset M$ then $\{f^*s_j\}$ is a frame for $\tilde{E}$ over $f^{-1}U$. We prove existence in three steps.

(i) First suppose $\tilde{M}$ is an open subset of $M$ and $f$ the inclusion map, so that $\tilde{E}$ is just the restriction of $E$ to $\tilde{M}$. In this case the existence of $\tilde{\nabla}$ follows from Proposition 3.1 and Corollary 3.1.

(ii) Next we consider the case when $E$ is trivial. Let $(s_1, \ldots, s_r)$ be a global frame for $E$ and $(\omega^i_j)$ the matrix of 1–forms on $M$ with

$$\nabla_Y s_j = \sum_i \omega^i_j(Y)s_i$$
for vector fields $Y$ on $M$. Set $\tilde{s}_j := f^*s_j$ and $\tilde{\omega}_j := f^*\omega_j$, and let $\tilde{\nabla}$ be the unique connection in $\tilde{E}$ such that

$$\tilde{\nabla}_X\tilde{s}_j = \sum_i \tilde{\omega}_i^j(X)\tilde{s}_i$$

for vector fields $X$ on $\tilde{M}$. We now verify that $\tilde{\nabla}$ satisfies (4). Let $p \in \tilde{M}$, $v \in T_p\tilde{M}$, and $s \in \Gamma(E)$. Then $s = \sum_j h^j s_j$ for some functions $h^j \in \mathcal{D}(M, \mathbb{K})$. Set $q := f(p)$, $w := f_*v$, and $g^j := h^j \circ f$. Then $f^*s = \sum_j g^j \tilde{s}_j$, and

$$\tilde{\nabla}_v\tilde{s}_j = \sum_i \tilde{\omega}_i^j(v)\tilde{s}_i(p) = \sum_i \omega_i^j(w) s_i(q) = \nabla_w s_j,$$

so

$$\tilde{\nabla}_v(f^*s) = \sum_j [v(g^j) \cdot \tilde{s}_j(p) + g^j(p) \cdot \tilde{\nabla}_v\tilde{s}_j]$$

$$= \sum_j [w(h^j) \cdot s_j(q) + h^j(q) \cdot \nabla_w s_j]$$

$$= \nabla_w s.$$

(iii) In the general case, choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of $M$ such that $E|_{U_\alpha}$ is trivial for each $\alpha$. Set $\tilde{U}_\alpha := f^{-1}U_\alpha$. By case (ii), we have a pull-back connection $\tilde{\nabla}^\alpha$ in $\tilde{E}|_{\tilde{U}_\alpha}$. By uniqueness, $\tilde{\nabla}^\alpha$ and $\tilde{\nabla}^\beta$ restrict to the same connection on $\tilde{U}_\alpha \cap \tilde{U}_\beta$ for each $\alpha, \beta \in I$. Hence, the connections $\tilde{\nabla}^\alpha$ patch together to give the desired connection in $\tilde{E}$.  

Example Let $E \to M$ be a $\mathbb{K}$–vector bundle with connection $\nabla$ and

$$\gamma : I \to M, \quad t \mapsto \gamma(t)$$

a smooth path in $M$, where $I \subset \mathbb{R}$ is an interval. Proposition 4.1 provides a $\mathbb{K}$–linear operator $\frac{D}{dt} := (\gamma^*\nabla)_{\frac{d}{dt}}$ acting on sections of $E$ along $\gamma$ with the following properties.

- For all smooth functions $f : I \to \mathbb{K}$ and sections $\sigma$ of $E$ along $\gamma$ one has
  $$\frac{D}{dt}(f\sigma) = \frac{df}{dt} \cdot \sigma + f \cdot \frac{D\sigma}{dt}.$$

- If $\sigma = f^*s$ is the pull-back of a section $s$ of $E$ then
  $$\frac{D\sigma}{dt} = \nabla_{\frac{d}{dt}}(s).$$
5 Holonomy

**Proposition 5.1** Let \( E \to M \) be a \( \mathbb{K} \)-vector bundle with connection \( \nabla \), and let \( \gamma : I \to M \) be a smooth curve. Let \( t_0 \in I \) and \( v \in E_{\gamma(t_0)} \). Then there exists a unique parallel section \( \sigma \) of \( E \) along \( \gamma \) such that \( \sigma(t_0) = v \).

Here, \( \sigma \) is called parallel if \( \frac{D\sigma}{dt} = 0 \).

**Proof.** It is a simple exercise to show that any vector bundle over an interval is trivial. Hence, there exists a global frame \((\sigma_1, \ldots, \sigma_r)\) of the pullback bundle \( \gamma^*E \). In terms of this frame, the operator \( \frac{D}{dt} \) is given by an \( r \times r \) matrix \((c_{ij})\) of functions \( I \to \mathbb{K} \) such that

\[
\frac{D\sigma_j}{dt} = \sum_i c_{ij} \sigma_i.
\]

We are therefore seeking smooth \( \mathbb{K} \)-valued functions \( f^1, \ldots, f^r \) on the interval \( I \) with specified values at \( t_0 \) such that

\[
0 = \frac{D}{dt} \sum_j f^j \sigma_j = \sum_j \left( \frac{df^j}{dt} \cdot \sigma_j + f^j \sum_i c_{ij} \sigma_i \right)
\]

\[
= \sum_i \left( \frac{df^i}{dt} + \sum_j c_{ij} f^j \right) \sigma_i,
\]

or, equivalently,

\[
\frac{df^i}{dt} + \sum_j c_{ij} f^j = 0, \quad i = 1, \ldots, r. \tag{5}
\]

Because this is a linear system of ordinary differential equations, it has a unique solution with given values at \( t_0 \). \( \square \)

The proposition allows us to define for any \( a, b \in I \) the **holonomy map**, or parallel transport,

\[
\text{Hol}_{a}^{b} := \text{Hol}_{a}^{b}(\nabla, \gamma) : E_{\gamma(a)} \to E_{\gamma(b)}
\]

of \( \nabla \) along \( \gamma \). This is the linear isomorphism characterized by the property that if \( \sigma \) is any parallel section of \( E \) along \( \gamma \) then

\[
\text{Hol}_{a}^{b}(\sigma(a)) = \sigma(b).
\]

Clearly, if \( a, b, c \in I \) then

\[
\text{Hol}_{a}^{c} = \text{Hol}_{b}^{c} \circ \text{Hol}_{a}^{b}.
\]
We can exploit this property to define $\text{Hol}_a^b$ even if $\gamma$ is only piecewise smooth. Namely, if 
\[ a = a_0 < a_1 < \cdots < a_n = b \]
and $\gamma_{[a_{i-1}, a_i]}$ is smooth for $i = 1, \ldots, n$ we define $\text{Hol}_a^b$ to be the composite of the holonomies along each subinterval, i.e.
\[ \text{Hol}_a^b := \text{Hol}_{a_n}^{a_{n-1}} \circ \cdots \circ \text{Hol}_{a_2}^{a_1} \circ \text{Hol}_{a_1}^{a_0}. \]

The following property expresses a connection in terms of its holonomy.

**Proposition 5.2** Let $\nabla$ be a connection in a vector bundle $E \to M$ and $s$ a section of $E$. Let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve, where $\epsilon > 0$. For $-\epsilon < t < \epsilon$ set 
\[ h_t := \text{Hol}_0^t(\nabla, \gamma) : E_{\gamma(t)} \to E_{\gamma(0)}. \]

Then
\[ \nabla_{\gamma(0)}(s) = \left. \frac{d}{dt} \right|_0 h_t(s(\gamma(t))). \]

Note that $t \mapsto h_t(s(\gamma(t)))$ is a curve in the finite-dimensional vector space $E_{\gamma(0)}$, and we are differentiating this curve at 0 in the usual sense.

**Proof.** Let $(\sigma_1, \ldots, \sigma_r)$ be a global frame for $\gamma^*E$ consisting of parallel sections. Then $\gamma^*s = \sum_j f^j \sigma_j$ for some functions $f^j : (-\epsilon, \epsilon) \to \mathbb{K}$. Now,
\[ h_t(s(\gamma(t))) = \sum_j f^j(t) \cdot \sigma_j(0), \]
and
\[ \nabla_{\gamma(0)}(s) = \left. \frac{D\gamma^*s}{dt} \right|_0(0) = \sum_j (f^j)'(0) \cdot \sigma_j(0) = \left. \frac{d}{dt} \right|_0 h_t(s(\gamma(t))). \]

**Proposition 5.3** Let $\nabla$ be a connection in a vector bundle $E$ over $M$, and $\mathcal{F}$ a smooth functor. Then there is a unique connection $\tilde{\nabla}$ in $\mathcal{F}(E)$ such that for any path $\gamma : I \to M$ and $a, b \in I$ one has
\[ \text{Hol}_a^b(\tilde{\nabla}, \gamma) = \mathcal{F}(\text{Hol}_a^b(\nabla, \gamma)) : \mathcal{F}(E_{\gamma(a)}) \to \mathcal{F}(E_{\gamma(b)}). \]

**Proof.** The uniqueness of such a connection $\tilde{\nabla}$ follows from Proposition 5.2, which expresses a connection in terms of its holonomy. To prove existence, it then suffices to consider the case of a product bundle $E = M \times V$. Let $a \in \Omega^1(M; \text{gl}(V))$ be the connection form of $\nabla$, so that $\nabla = d + a$, where
\[ d \] is the product connection. Recall that the functor \( \mathcal{F} \) gives defines a Lie group homomorphism
\[ f : \text{GL}(V) \to \text{GL}(\tilde{V}), \]
where \( \tilde{V} := \mathcal{F}(V) \). Let \( L : \text{gl}(V) \to \text{gl}(\tilde{V}) \) be corresponding Lie algebra homomorphism, i.e. the derivative of \( f \) at the identity. We define \( \tilde{\nabla} \) to be the connection in the product bundle \( M \times \tilde{V} \) over \( M \) with connection form \( L_a \in \Omega^1(M; \text{gl}(\tilde{V})) \). To show that \( \tilde{\nabla} \) has the desired property, let \( \gamma : I \to M \) be a curve. Define \( c : I \to \text{gl}(V) \) by
\[ c(t) := a(\gamma'(t)). \]
If \( s \) is any section of \( E \) along \( \gamma \), thought of as a map \( I \to V \), then
\[ \frac{Ds}{dt} = s' + cs, \]
where \( s' \) is the usual derivative of \( s \). Now fix \( t_0 \in I \) and define \( u : I \to \text{GL}(V) \) by
\[ u(t) := \text{Hol}^t_{t_0}(\nabla, \gamma). \]
For any \( v \in V \) the map \( t \mapsto u(t)v \) defines a parallel section of \( E \) along \( \gamma \), hence
\[ 0 = \frac{D(uv)}{dt} = u'v + cuv. \]
Therefore, \( u \) is unique solution to the equations
\[ u' + cu = 0, \quad u(t_0) = \text{Id}_V. \]
We claim that the map
\[ \tilde{u} := f \circ u : I \to \text{GL}(\tilde{V}) \]
satisfies a similar equation. Namely,
\[ \tilde{u}'(t) = f_*(u'(t)) = -f_*(c(t)u(t)) = -\left. \frac{d}{ds} \right|_0 f(\exp(sc(t)) \cdot u(t)) \]
\[ = -\left. \frac{d}{ds} \right|_0 f(\exp(sc(t)) \cdot \tilde{u}(t)) = -Lc(t)\tilde{u}(t). \]
Because \( \tilde{u}(t_0) = \text{Id}_{\tilde{V}} \), we conclude that
\[ \tilde{u}(t) = \text{Hol}^t_{t_0}(\tilde{\nabla}, \gamma). \]
This proves the existence of the desired connection in \( \mathcal{F}(E) \) in the case when \( E \) is trivial, hence also in general. \( \square \)
Proposition 5.4 Let $\nabla$ be the connection in $\Lambda M$ induced by a given connection in the tangent bundle $TM$. Then for any vector field $X$ on $M$ and differential forms $\alpha, \beta$ on $M$ one has

$$\nabla_X(\alpha \wedge \beta) = \nabla_X \alpha \wedge \beta + \alpha \wedge \nabla_X \beta.$$ 

Here, one could replace the tangent bundle with any vector bundle over $M$, but we are mainly interested in $TM$.

Proof. Let $\nabla'$ be the given connection in $TM$ and $\gamma : (-\epsilon, \epsilon) \to M$ a smooth curve. Let $p := \gamma(0)$ and $v := \gamma'(0)$ and for $-\epsilon < t < \epsilon$ set

$$u_t := \text{Hol}_t^\gamma(\nabla', \gamma),$$

so that $u_t = (h_t)^{-1}$ in the notation of Proposition 5.2. Then

$$\nabla_v(\alpha \wedge \beta) = \left. \frac{d}{dt} \right|_0 u_t^*(\alpha_{\gamma(t)} \wedge \beta_{\gamma(t)})$$

$$= \left. \frac{d}{dt} \right|_0 (u_t^* \alpha_{\gamma(t)} \wedge u_t^* \beta_{\gamma(t)})$$

$$= \left( \left. \frac{d}{dt} \right|_0 u_t^* \alpha_{\gamma(t)} \right) \wedge \beta_p + \alpha_p \wedge \left( \left. \frac{d}{dt} \right|_0 u_t^* \beta_{\gamma(t)} \right)$$

$$= \nabla_v \alpha \wedge \beta_p + \alpha_p \wedge \nabla_v \beta. \quad \square$$

Proposition 5.3 can be generalized to smooth functors of several variables such as the tensor product. Given vector bundles $E$ and $E'$ over $M$, equipped with connections $\nabla$ and $\nabla'$, respectively, we obtain a connection $\tilde{\nabla}$ in $E \otimes E'$ which is uniquely characterized by the fact that for any vector field $X$ on $M$ and sections $s, s'$ of $E, E'$ one has

$$\tilde{\nabla}_X(s \otimes s') = \nabla_X s \otimes s' + s \otimes \nabla'_X s'.$$

Such a connection can also be constructed using local frames for $E$ and $E'$.

6 Curvature

The curvature $F = F^\nabla$ of a connection $\nabla$ in a $\mathbb{K}$-vector bundle $E \to M$ associates to any pair $X, Y$ of vector fields on $M$ the map $\Gamma(E) \to \Gamma(E)$ given by

$$F(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s.$$
If \( F(X, Y) = 0 \) for all \( X, Y \) then \( \nabla \) is called flat. We will sometimes use the notation
\[
[\nabla_X, \nabla_Y] := \nabla_X \nabla_Y - \nabla_Y \nabla_X.
\]

**Example** Let \( \nabla \) be the product connection in \( E := M \times \mathbb{K}^r \to M \). Identifying sections of \( E \) with functions \( h : M \to \mathbb{K}^r \) we have \( \nabla_X h = Xh \), so
\[
\nabla_X \nabla_Y h - \nabla_Y \nabla_X h = XYh - YXh = [X, Y]h = \nabla_{[X,Y]} h.
\]
Therefore, the product connection is flat.

**Proposition 6.1** Let \( \nabla \) be a connection in a vector bundle \( E \to M \). Then the map
\[
\mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, Y, s) \mapsto F(X, Y)s
\]
is \( \mathcal{D}(M, \mathbb{K}) \)-multilinear. The curvature of \( \nabla \) can therefore be regarded as a bundle-valued 2–form
\[
F \in \Omega^2(M; \text{End}(E)).
\]

**Proof.** We prove linearity in \( X \) (and hence in \( Y \) because of the skew-symmetry), leaving the linearity in \( s \) as an exercise for the reader.

Let \( g \in \mathcal{D}(M, \mathbb{K}) \). Then
\[
[gX, Y] = g[X, Y] - (Yg)X,
\]
so
\[
F(gX, Y) = \nabla_{gX} \nabla_Y s - \nabla_Y \nabla_{gX} s - \nabla_{[gX,Y]} s
= g\nabla_X \nabla_Y s - [(Yg)\nabla_X s + g\nabla_Y \nabla_X s] - [g\nabla_{[X,Y]} s - (Yg)\nabla_X s]
= gF(X, Y)s.
\]

It follows from the proposition that if \( E \) is trivial and \( (s_1, \ldots, s_r) \) is a global frame for \( E \) then there is an \( r \times r \) matrix \( (\Omega^i_j) \) of 2–forms on \( M \), called the curvature form of \( \nabla \) with respect to the given frame, such that for all vector fields \( X, Y \) on \( M \) one has
\[
F(X, Y)s_j = \sum_i \Omega^i_j(X, Y)s_i.
\]
**Theorem 6.1** Let $\nabla$ be a connection in a trivial vector bundle $E \to M$, and let $(s_1, \ldots, s_r)$ be a global frame for $E$. Let $\omega = (\omega^i_j)$ and $\Omega = (\Omega^i_j)$ be the corresponding connection form and curvature form of $\nabla$, respectively. Then for all $i, k$ one has

$$\Omega^i_k = d\omega^i_k + \sum_j \omega^i_j \wedge \omega^j_k.$$ 

In matrix notation, $$\Omega = d\omega + \omega \wedge \omega.$$

**Proof.** Recall that for any 1–forms $\alpha, \beta$ on $M$ one has

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

Now

$$\nabla_X \nabla_Y s_k = \sum_j \nabla_X (\omega^j_k(Y)s_j)$$

$$= \sum_j \left(X(\omega^j_k(Y)) \cdot s_j + \omega^j_k(Y) \sum_i \omega^i_j(X)s_i\right)$$

$$= \sum_i \left(X(\omega^i_k(Y)) + \sum_j \omega^i_j(X)\omega^j_k(Y)\right)s_i,$$

and of course there is a similar formula with $X, Y$ switched. Combining this with

$$\nabla_{[X, Y]} s_k = \sum_i \omega^i_k([X, Y])s_i$$

we obtain

$$F(X, Y)s_k = \sum_i \left(d\omega^i_k + \sum_j \omega^i_j \wedge \omega^j_k\right)(X, Y) \cdot s_i.$$ 

**Corollary 6.1** In the situation of the theorem, let $f : \tilde{M} \to M$ be a smooth map and $(\tilde{\Omega}^i_j)$ the curvature form of the pull-back connection $f^* \nabla$ with respect to the global frame $(f^* s_j)$ for $f^* E$. Then

$$\tilde{\Omega}^i_j = f^* \Omega^i_j.$$ 

\square
A connection $\nabla$ in a vector bundle $E \to M$ is called \textbf{trivial} if there exists a global frame $(s_j)$ for $E$ such that $\nabla s_j = 0$ for each $j$. The connection is called \textbf{locally trivial} if every point in $M$ has a neighbourhood over which $\nabla$ is trivial.

**Theorem 6.2** A connection is flat if and only if it is locally trivial.

\textit{Proof.} A locally trivial connection can locally be represented by the zero connection form and is therefore flat by Theorem 6.1. To prove the converse, let $\nabla$ be a flat connection in a trivial vector bundle $E \to \mathbb{R}^n$. We will construct a parallel section of $E$ with any prescribed value at the origin.

Let $(x^1, \ldots, x^n)$ be coordinates on $\mathbb{R}^n$ and $X_j := \frac{\partial}{\partial x^j}$. For $k = 0, \ldots, n$, let $V_k := \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ and $E_k := E|_{V_k}$. We first observe that for $k > 0$, any section $\sigma$ of $E_{k-1}$ can be extended to a section $\tau$ of $E_k$ by means of holonomy along lines parallel to the $k$th coordinate axis. Explicitly, $\tau$ is the unique section of $E_k$ such that

$$\nabla_{X_k} \tau = 0, \quad \tau|_{V_{k-1}} = \sigma. \quad (7)$$

To see that these equations have a unique smooth solution $\tau$, let $(s_1, \ldots, s_r)$ be a global frame for $E$ and

$$\nabla_{X_k} s_j = \sum_i c^i_j s_i,$$

where the coefficients $c^i_j$ are smooth functions on $\mathbb{R}^n$. If $\sigma = \sum_j f^j s_j$ and $\tau = \sum_j g^j s_j$, where $f^j$ and $g^j$ are functions on $V_{k-1}$ and $V_k$, respectively, then the equations (7) are equivalent to

$$\frac{\partial g^i}{\partial x^k} + \sum_j c^i_j g^j = 0, \quad g^i|_{V_{k-1}} = f^i \quad (i = 1, \ldots, r).$$

The theory of ordinary differential equations guarantees that these equations have a unique smooth solution $(g^i)$, see for instance [John Lee: Introduction to Smooth Manifolds, Theorem D.6].

We now prove that if $\sigma$ is parallel then so is $\tau$. In fact, for $1 \leq j < k$ we have

$$\nabla_{X_k} \nabla_{X_j} \tau = \nabla_{X_j} \nabla_{X_k} \tau = 0.$$

Because

$$\nabla_{X_j} \tau|_{V_{k-1}} = \nabla_{X_j} \sigma = 0$$

it follows that $\nabla_{X_j} \tau = 0$, so $\tau$ is parallel as claimed.

Applying the above procedure for $k = 1, \ldots, n$ we obtain a parallel section of $E$ with any prescribed value at the origin. \hfill \Box
7 The exterior covariant derivative

A connection $\nabla$ in a $K$–vector bundle $E \to M$ gives rise to a $K$–linear operator

$$d^{\nabla} : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$$

for $k \geq 0$, called the **exterior covariant derivative**. In degree zero, the operator $d^{\nabla}$ is just the connection $\nabla$ itself. To define it in arbitrary degrees, we use an auxiliary torsion-free connection $\nabla'$ in $TM$. Together, $\nabla$ and $\nabla'$ induce a connection $\nabla''$ in $\Lambda M \otimes E$, and we define $d^{\nabla}$ in degree $k$ to be the composite map

$$\Gamma(\Lambda^k M \otimes E) \xrightarrow{\nabla''} \Gamma(\Lambda^1 M \otimes \Lambda^k M \otimes E) \xrightarrow{\hat{\cdot}} \Gamma(\Lambda^{k+1} M \otimes E),$$

where the second map is given by the wedge product on forms. The following proposition provides an alternative formula for $d^{\nabla}$ which does not involve $\nabla'$.

**Proposition 7.1** For any $\omega \in \Omega^k(M; E)$ and vector fields $X_0, \ldots, X_k$ on $M$ one has

$$(d^{\nabla} \omega)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i} \left( \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \right)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).$$

Here, as usual, a hat over a term indicates that the term is to be omitted.

**Proof.** Let $(x^1, \ldots, x^n)$ be local coordinates on $M$ and set $\partial_j := \frac{\partial}{\partial x_j}$. In the coordinate domain we then have

$$d^{\nabla} \omega = \sum_{j=1}^{n} dx^j \wedge \nabla_{\partial_j} \omega,$$
which gives

\[
(d^\nabla \omega)(X_0, \ldots, X_k) = \sum_{j=1}^{n} \sum_{i=0}^{k} (-1)^i dx^i(X_i) (\nabla'_{X_0} \omega)(X_0, \ldots, \widehat{X}_i, \ldots, X_k)
\]

\[
= \sum_{i=0}^{k} (-1)^i (\nabla'_{X_i} \omega)(X_0, \ldots, \widehat{X}_i, \ldots, X_k)
\]

\[
= \sum_{i=0}^{k} (-1)^i \nabla X_i \left( \omega(X_0, \ldots, \widehat{X}_i, X_k) \right)
\]

\[
+ \sum_{i \neq j} (-1)^{i+j+1} \omega(X_0, \ldots, \widehat{X}_i, \ldots, \nabla'_{X_i} X_j, \ldots, X_k).
\]

Let \( S \) denote the sum on the last line. Then

\[
S = \sum_{i<j} (-1)^{i+j} \omega(X_0, \ldots, \widehat{X}_i, \ldots, \nabla'_{X_i} X_j, \ldots, X_k)
\]

\[
+ \sum_{j<i} (-1)^{i+j} \omega(X_0, \ldots, \nabla'_{X_j} X_i, \ldots, X_k)
\]

\[
= \sum_{i<j} (-1)^{i+j} \omega(\nabla'_{X_i} X_j, X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k)
\]

\[
+ \sum_{j<i} (-1)^{i+j+1} \omega(\nabla'_{X_i} X_j, X_0, \ldots, \widehat{X}_j, \ldots, \widehat{X}_i, \ldots, X_k).
\]

Interchanging \( i \) and \( j \) in the last sum and recalling that

\[
\nabla'_{X_i} X_j - \nabla'_{X_j} X_i = [X_i, X_j]
\]

since \( \nabla' \) is torsion-free we obtain

\[
S = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k).
\]

For \( \alpha \in \Omega^1(M; E) \) the proposition says that

\[
(d^\nabla \alpha)(X, Y) = \nabla_X (\alpha(Y)) - \nabla_Y (\alpha(X)) - \alpha([X, Y]).
\]

Proposition 7.2 If \( \nabla \) is the product connection in \( M \times \mathbb{R} \) then \( d^\nabla : \Omega^k(M) \rightarrow \Omega^{k+1}(M) \) coincides with the exterior derivative \( d \).
Proof. This is a local statement, so we may assume $M = \mathbb{R}^n$. Since $d$ and $d\bar{\nabla}$ are both $\mathbb{R}$-linear it suffices to show that they agree on forms $\omega = f \, dx^I$, where $f \in \mathcal{D}(\mathbb{R}^n)$ and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ for some increasing multi-index $I = (i_1, \ldots, i_k)$. Taking $\nabla'$ to be the product connection in $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ we have $\nabla'(dx^I) = 0$. Setting $X_j := \frac{\partial}{\partial x^j}$ we obtain
\[
d\bar{\nabla}\omega = \sum_j dx^j \wedge \nabla X_j \omega \]
\[
= \sum_j dx^j \wedge (X_j f) dx^I \]
\[
= df \wedge dx^I \]
\[
= d\omega.
\]

Proposition 7.3 Let $\nabla$ be a connection in a vector bundle $E \to M$. Then for any $\alpha \in \Omega^k(M)$ and $\phi \in \Omega^\ell(M; E)$ one has
\[
d\nabla (\alpha \wedge \phi) = d\alpha \wedge \phi + (-1)^k \alpha \wedge d\nabla \phi.
\]

Note that this property, together with the fact that $d\bar{\nabla}s = \nabla s$ for $s \in \Omega^0(M; E) = \Gamma(E)$, determines $d\bar{\nabla}$ completely.

Proof. Let $E_1, \ldots, E_n$ be a frame of vector fields on an open subset $U \subset M$ and $\eta^1, \ldots, \eta^n$ the dual frame of 1–forms. Then over $U$ one has
\[
d\nabla (\alpha \wedge \phi) = \sum_j \eta^j \wedge \nabla'_{E_j} (\alpha \wedge \phi) \]
\[
= \sum_j \eta^j \wedge (\nabla'_{E_j} \alpha \wedge \phi + \alpha \wedge \nabla''_{E_j} \phi) \]
\[
= \left( \sum_j \eta^j \wedge \nabla'_{E_j} \alpha \right) \wedge \phi + (-1)^k \alpha \wedge \sum_j \eta^j \wedge \nabla''_{E_j} \phi \]
\[
= d\alpha \wedge \phi + (-1)^k \alpha \wedge d\nabla \phi. \]

Theorem 7.1 Let $\nabla$ be a connection in a vector bundle $E \to M$, and let $F$ be the curvature of $\nabla$. Then for all $\omega \in \Omega^k(M; E)$ one has
\[
d\nabla (d\nabla(\omega)) = F \wedge \omega.
\]

Here, the pairing $F \wedge \omega$ combines the wedge product $\Lambda^2 M \otimes \Lambda^k M \to \Lambda^{k+2} M$ with the evaluation map $\text{End}(E) \otimes E \to E$.
Proof. Since the statement is local and both sides of the equation are \( \mathbb{R} \)-linear in \( \omega \) we may assume \( \omega = \alpha \otimes s \) for some \( \alpha \in \Omega^k(M) \) and \( s \in \Gamma(E) \). For vector fields \( X, Y \) on \( M \), Equation 8 yields

\[
(d\nabla d\nabla s)(X,Y) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, 
\]

i.e. \( d\nabla d\nabla s = Fs \). Applying Proposition 7.3 twice we get

\[
d\nabla d\nabla \omega = d\nabla (d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s) = \alpha \wedge Fs = F \wedge \omega. \]

Theorem 7.2 (Bianchi) Let \( \nabla \) be a connection in \( E \to M \) with curvature \( F \). Let \( \bar{\nabla} \) denote the induced connection in \( \text{End}(E) \). Then

\[
d\bar{\nabla} F = 0. \]

Proof. For all sections \( s \) of \( E \) one has

\[
F \wedge \nabla s = d\nabla (d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s) = \alpha \wedge Fs = F \wedge \nabla s, 
\]

hence \( (d\bar{\nabla} F)s = 0. \)

8 Orthogonal connections

By a Euclidean metric on a real vector bundle \( E \to M \) we mean a choice of scalar product \( \langle \cdot, \cdot \rangle \) on each fibre \( E_p \) such that for every pair \( s, t \) of sections of \( E \) the function

\[
M \to \mathbb{R}, \quad p \mapsto \langle s(p), t(p) \rangle
\]

is smooth. A real vector bundle equipped with a Euclidean metric is called a Euclidean vector bundle.

Example The product bundle \( M \times \mathbb{R}^k \to M \) where each fibre has the standard Euclidean metric.

Theorem 8.1 Every real vector bundle admits a Euclidean metric.

Proof. Glue together local Euclidean metrics using a partition of unity. 

Let \( E \to M \) be a Euclidean vector bundle of rank \( k \). By an orthonormal frame for \( E \) over an open set \( U \subset M \) we mean a \( k \)-tuple \((s_1, \ldots, s_k)\) of sections of \( E_{|U} \) such that \((s_1(p), \ldots, s_k(p))\) is an orthonormal basis for \( E_p \) for every \( p \in U \). Note that the Gram-Schmidt process applied pointwise to an arbitrary frame for \( E \) over \( U \) yields an orthonormal frame.
Let $E \rightarrow M$ be a real vector bundle equipped with a connection $\nabla$ as well as a Euclidean metric. Then $\nabla$ is called **orthogonal**, or **compatible with the Euclidean metric**, if for all sections $s, t$ of $E$ and vector fields $X$ on $M$ one has

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle.$$ 

**Example** If each fibre of a product bundle $E = M \times \mathbb{R}^k \rightarrow M$ has the same scalar product, then the product connection in $E$ is orthogonal.

**Theorem 8.2** Every Euclidean vector bundle admits an orthogonal connection.

*Proof.* This is proved just as Theorem 3.1. □

For any finite-dimensional Euclidean space $V$ let $\text{so}(V)$ denote the Lie algebra of skew-symmetric linear endomorphisms of $V$. If $E \rightarrow M$ is a Euclidean vector bundle then we can apply this construction fibrewise to obtain a subbundle

$$\text{so}(E) := \bigcup_{p \in M} \text{so}(E_p)$$

of the endomorphism bundle $\text{End}(E)$.

**Proposition 8.1** Let $\nabla$ be an orthogonal connection in a Euclidean vector bundle $E \rightarrow M$, and $a \in \Omega^1(M; \text{End}(E))$. Then the connection $\nabla + a$ is orthogonal if and only if $a$ takes values in $\text{so}(E)$.

*Proof.* Set $\nabla' := \nabla + a$. Then for any vector field $X$ on $M$ and sections $s, t$ of $E$ one has

$$\langle \nabla'_X s, t \rangle + \langle s, \nabla'_X t \rangle = X\langle s, t \rangle + \langle a(X)s, t \rangle + \langle s, a(X)t \rangle.$$ 

From this the proposition follows immediately. □

**Theorem 8.3** Let $E \rightarrow M$ be a real vector bundle equipped with both a connection $\nabla$ and a Euclidean metric. Then the following are equivalent.

(i) $\nabla$ is orthogonal.

(ii) For every piecewise smooth curve $\gamma : [a, b] \rightarrow M$ the holonomy $E_{\gamma(a)} \rightarrow E_{\gamma(b)}$ of $\nabla$ along $\gamma$ is an orthogonal linear map.
Proof. Suppose $\nabla$ is orthogonal and let $\gamma: [a, b] \to M$ be a piecewise smooth curve. We will show that the holonomy of $\nabla$ along $\gamma$ is orthogonal. Because of the composition law (6) for the holonomy we may assume $\gamma$ is smooth. We may in fact also assume $E$ is trivial, since for any sufficiently fine partition of the interval $[a, b]$ the image of each subinterval under $\gamma$ will be contained in an open subset of $M$ over which $E$ is trivial.

Let $(\omega_j^i)$ be the connection form of $\nabla$ with respect to a global orthonormal frame $(s_1, \ldots, s_k)$ for $E$. By Proposition 8.1, the connection form is skew-symmetric. To see this explicitly, let $X$ be a vector field on $M$. Then

\[ 0 = X\langle s_i, s_j \rangle = \langle \nabla_X s_i, s_j \rangle + \langle s_i, \nabla_X s_j \rangle = \sum_k \omega_k^i (X) s_k \langle s_i, s_j \rangle + \langle s_i, \sum_k \omega_k^j (X) s_k \rangle = \omega_j^i (X) + \omega_i^j (X), \]

so that $\omega_j^i = -\omega_i^j$. Now let $\sigma$ be a parallel section of $E$ along $\gamma$. We will show that the pointwise norm $|\sigma|$ is a constant function on $[a, b]$, which implies that the holonomy of $\nabla$ along $\gamma$ is orthogonal. Since $(\gamma^* s_i)$ is a global orthonormal frame for $\gamma^* E$ we have $\sigma = \sum_i f^i \gamma^* s_i$ for some real-valued functions $f^i$ on the interval $[a, b]$, and

\[ |\sigma|^2 = \sum_i (f^i)^2. \]

Let $c_j^i$ be the functions given by $\gamma^* \omega_j^i = c_j^i dt$. Equation (5) now yields

\[ \frac{d}{dt}|\sigma(t)|^2 = 2 \sum_i f^i \cdot \frac{df^i}{dt} = -2 \sum_{ij} f^i f^j c_j^i = 0 \]

by the skew-symmetry of the matrix $(c_j^i)$. Hence, $\sigma$ has constant length as claimed.

To prove the reverse implication in the proposition, suppose (ii) holds and let $s_1, s_2$ be a pair of sections of $E$ and $X$ a vector field on $M$. Given a point $p \in M$ we can find $\epsilon > 0$ and a smooth curve $\gamma: (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. For $-\epsilon < t < \epsilon$ let

\[ h_t = \text{Hol}_t^0 (\nabla; \gamma): E_{\gamma(t)} \to E_p \]

24
be the holonomy of $\nabla$, which is orthogonal by assumption. Using Proposition 5.2 we obtain

\[
X_p\langle s_1, s_2 \rangle = \frac{d}{dt} \bigg|_0 \langle s_1(\gamma(t)), s_2(\gamma(t)) \rangle \\
= \frac{d}{dt} \bigg|_0 \langle h_t s_1(\gamma(t)), h_t s_2(\gamma(t)) \rangle \\
= \left\langle \frac{d}{dt} \bigg|_0 h_t s_1(\gamma(t)), s_2(p) \right\rangle + \left\langle s_1(p), \frac{d}{dt} \bigg|_0 h_t s_2(\gamma(t)) \right\rangle \\
= \langle \nabla_X s_1, s_2 \rangle_p + \langle s_1, \nabla_X s_2 \rangle_p,
\]

which shows that $\nabla$ is orthogonal. □

**Proposition 8.2** Let $\nabla$ be an orthogonal connection in a Euclidean vector bundle $E \to M$ and $F$ the curvature of $\nabla$. Then $F(X,Y)$ is a section of $\text{so}(E)$ for all vector fields $X, Y$ on $M$, i.e.

$F \in \Omega^2(M; \text{so}(E))$.

**Proof.** For any sections $s, t$ of $E$ we have

\[
XY\langle s, t \rangle = \langle \nabla_X \nabla_Y s, t \rangle + \langle \nabla_Y s, \nabla_X t \rangle + \langle \nabla_X s, \nabla_Y t \rangle + \langle s, \nabla_X \nabla_Y t \rangle.
\]

Combining this with the corresponding equality with $X$ and $Y$ interchanged we obtain

\[
\langle [\nabla_X, \nabla_Y]s, t \rangle + \langle s, [\nabla_X, \nabla_Y]t \rangle = [X, Y]\langle s, t \rangle \\
= \langle \nabla_{[X,Y]}s, t \rangle + \langle s, \nabla_{[X,Y]}t \rangle.
\]

Therefore,

\[
\langle F(X,Y)s, t \rangle + \langle s, F(X,Y)t \rangle = 0. \quad \Box
\]

A **Riemannian manifold** is a smooth manifold $M$ equipped with a Euclidean metric in $TM$.

**Theorem 8.4 (Levi-Civita)** If $M$ is a Riemannian manifold then $TM$ has a unique torsion-free orthogonal connection.

The proof can be found in any book on Riemannian geometry.