MAT4640 - 2014

Mandatory assignment

Deadline: March 20 - 14:30

The solution may be submitted through the standard procedure or in written or electronic form to Dag Normann, dnormann@math.uio.no.
Scanned copies of handwritten solutions will not be accepted if they are hard to read.

Problem 1

Express the formula

$$x = y \cup \{y\}$$

using x, y, $\in$ and $=$, standard logical connectives and quantifiers, but no defined functions or predicates.

Problem 2

Let ω_1 be the first uncountable ordinal, and let $f : \omega_1 \to \omega_1$ be such that

- If $\alpha < \beta < \omega_1$ then $f(\alpha) < f(\beta)$.
- If α is a limit ordinal, then

$$f(\alpha) = \bigcup_{\beta<\alpha} f(\beta).$$

a) Prove that $\alpha \leq f(\alpha)$ for all $\alpha < \omega_1$.

b) Prove that $f(\alpha) = \alpha$ for uncountably many $\alpha < \omega_1$.

Problem 3

Let X be the set of functions

$$f : \omega \to \omega_1$$

where ω_1 is the first uncountable cardinal.
Show that

$$X \approx 2^\omega$$

i.e., they have the same cardinality.
Discuss where you use the axiom of choice.

Problem 4

Show that there is one, and only one, function $f : R(\omega) \to \omega$ satisfying

$$f(x) = \sum_{y \in x} 2^{f(y)}$$

for each $x \in R(\omega)$. Here $R(\omega)$ is the first infinite level in the cumulative hierarchy.
Show that this function is a bijection.
Find a function $g : \omega \to \omega$ such that for all n and m in ω and for x and y in $R(\omega)$ with $n = f(x)$ and $m = f(y)$ we have that

$$f(\langle x, y \rangle) = g(n, m).$$

Problem 5

Let α and β be two limit ordinals. Show that 1. $\Rightarrow$ 2. where

1. $cf(\alpha) < cf(\beta)$
2. Every increasing $h : \alpha \to \beta$ has a range that is bounded in β.

Note that we do not restrict 2. to *strictly* increasinng functions.
Does the implication hold if we do not even demand h to be increasing?

END OF ASSIGNMENT