The proof of the Schröder-Bernstein theorem

Since there was some confusion in the presentation of the proof of this theorem on February 5, I offer some details here.

**Theorem 1** If \( f : A \to B \) and \( g : B \to A \) are two injective functions, there is a bijection \( h \) from \( A \) to \( B \).

**Proof**

Let \( A_0 = A \) and \( B_0 = B \).

By recursion, let \( B_{n+1} = f[A_n] \) and \( A_{n+1} = A \setminus g[B \setminus B_{n+1}] \)

(Here \( f[A_n] \) etc. denotes the range of \( A_n \) under \( f \)).

This definition is slightly different from the one given at the lecture, but will serve the same purpose.

By induction, we see that \( B_{n+1} \subseteq B_n \) and that \( A_{n+1} \subseteq A_n \).

Let \( A_\omega = \bigcap_{n \in \omega} A_n \) and \( B_\omega = \bigcap_{n \in \omega} B_n \).

We will prove

1. \( f \) restricted to \( A_\omega \) is a bijection to \( B_\omega \)
2. \( g \) restricted to \( B \setminus B_\omega \) is a bijection to \( A \setminus A_\omega \).

Then we may define

\[
h(a) = \begin{cases} 
  f(a) & \text{if } a \in A_\omega \\
  g^{-1}(a) & \text{if } a \notin A_\omega,
\end{cases}
\]

and \( h \) will be a bijection.

If \( a \in A_\omega \), then \( a \in A_n \) for all \( n \), so \( f(a) \in B_{n+1} \) for all \( n \) and \( f(a) \in B_\omega \).

Conversely, if \( b \in B_\omega \), then \( b \in B_{n+1} \) for all \( n \), so for all \( n \) there is an \( a_n \in A_n \) such that \( b = f(a_n) \). Since \( f \) is injective, all the \( a_n \)'s are equal, call the value \( a \). Then \( a \in A_\omega \) and \( b = f(a) \). This proves that \( f \) is a bijection from \( A_\omega \) to \( B_\omega \).

Now, let \( b \notin B_\omega \). Then there is an \( n \in \omega \) such that \( b \notin B_{n+1} \). By the definition of \( A_{n+1} \) we have that \( g(b) \notin A_{n+1} \) so \( g(b) \notin A_\omega \).

Conversely, if \( a \notin A_\omega \) there is some \( n \) such that \( a \notin A_{n+1} \), and the reason must be that there is some \( b \notin B_{n+1} \) such that \( g(b) = a \). This \( b \) will be unique, and will not be in \( B_\omega \).

This proves that \( g \) is a bijection from the complement of \( B_\omega \) to the complement of \( A_\omega \), so the second claim is proved.