The proof of the Schröder-Bernstein theorem

Since there was some confusion in the presentation of the proof of this theorem on February 5, I offer some details here.

Theorem 1 *If* $f : A \to B$ *and* $g : B \to A$ *are two injective functions, there is a bijection* h *from* A *to* B*.*

Proof
Let $A_0 = A$ and $B_0 = B$.

By recursion, let $B_{n+1} = f[A_n]$ and $A_{n+1} = A \setminus g[B \setminus B_{n+1}]$
(Here $f[A_n]$ etc. denotes the range of A_n under f).

This definition is slightly different from the one given at the lecture, but will serve the same purpose.
By induction, we see that $B_{n+1} \subseteq B_n$ and that $A_{n+1} \subseteq A_n$.

Let $A_\omega = \bigcap_{n\in\omega} A_n$ and $B_\omega = \bigcap_{n\in\omega} B_n$.
We will prove

1. f restricted to A_ω is a bijection to B_ω
2. g restricted to $B \setminus B_\omega$ is a bijection to $A \setminus A_\omega$.

Then we may define

$$h(a) = \begin{cases} f(a) & \text{if} \quad a \in A_\omega \\ g^{-1}(a) & \text{if} \quad a \notin A_\omega, \end{cases}$$

and h will be a bijection.

If $a \in A_\omega$, then $a \in A_n$ for all n, so $f(a) \in B_{n+1}$ for all n and $f(a) \in B_\omega$. Conversely, if $b \in B_\omega$, then $b \in B_{n+1}$ for all n, so for all n there is an $a_n \in A_n$ such that $b = f(a_n)$. Since f is injective, all the a_n's are equal, call the value a. Then $a \in A_\omega$ and $b = f(a)$. This proves that f is a bijection from A_ω to B_ω.

Now, let $b \notin B_\omega$. Then there is an $n \in \omega$ such that $b \notin B_{n+1}$. By the definition of A_{n+1} we have that $g(b) \notin A_{n+1}$ so $g(b) \notin A_\omega$.
Conversely, if $a \notin A_\omega$ there is some n such that $a \notin A_{n+1}$, and the reason must be that there is some $b \notin B_{n+1}$ such that $g(b) = a$. This b will be unique, and will not be in B_ω.
This proves that g is a bijection from the complement of B_ω to the complement of A_ω, so the second claim is proved.